Gentle introduction to abstract interpretation

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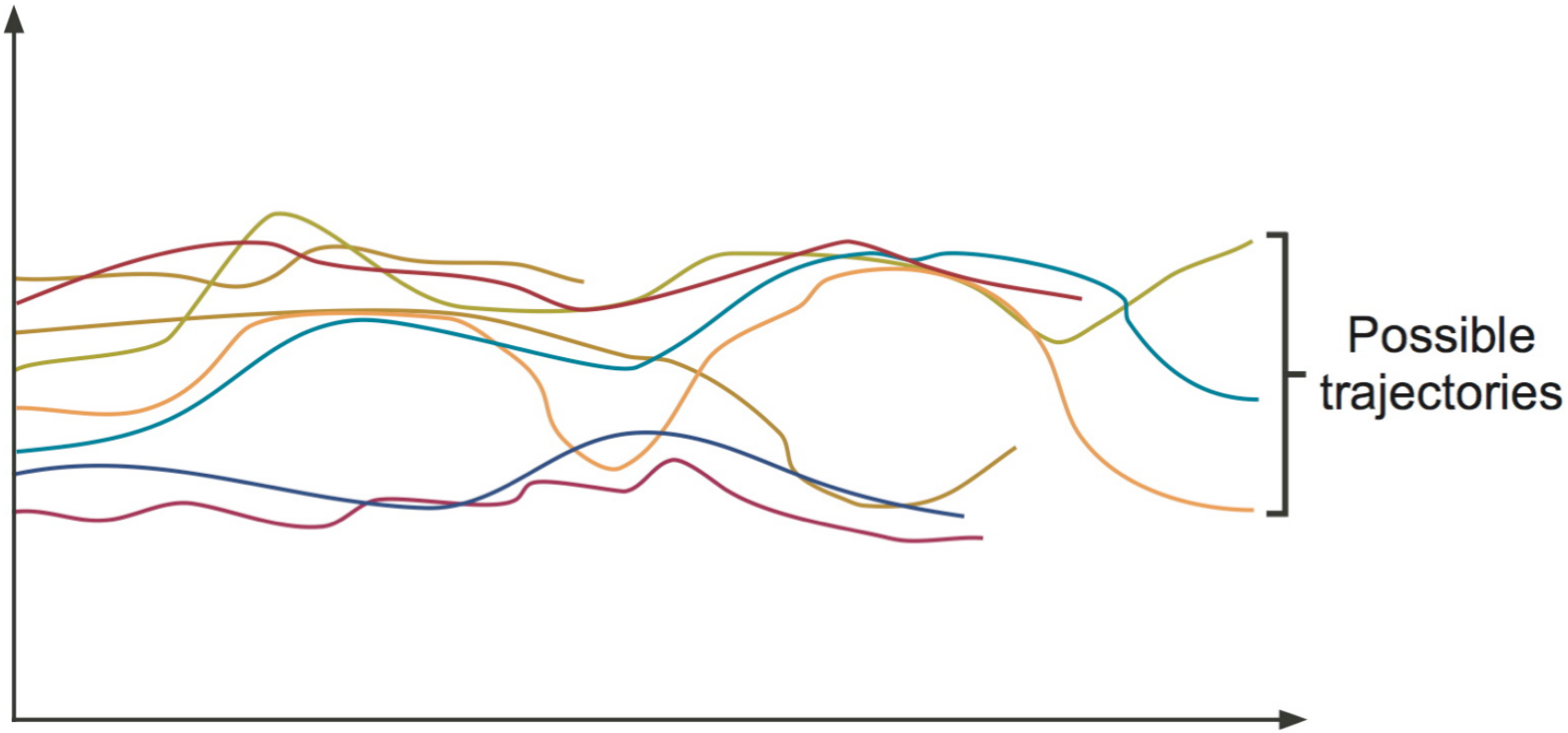
September 25, 2017

Motivation

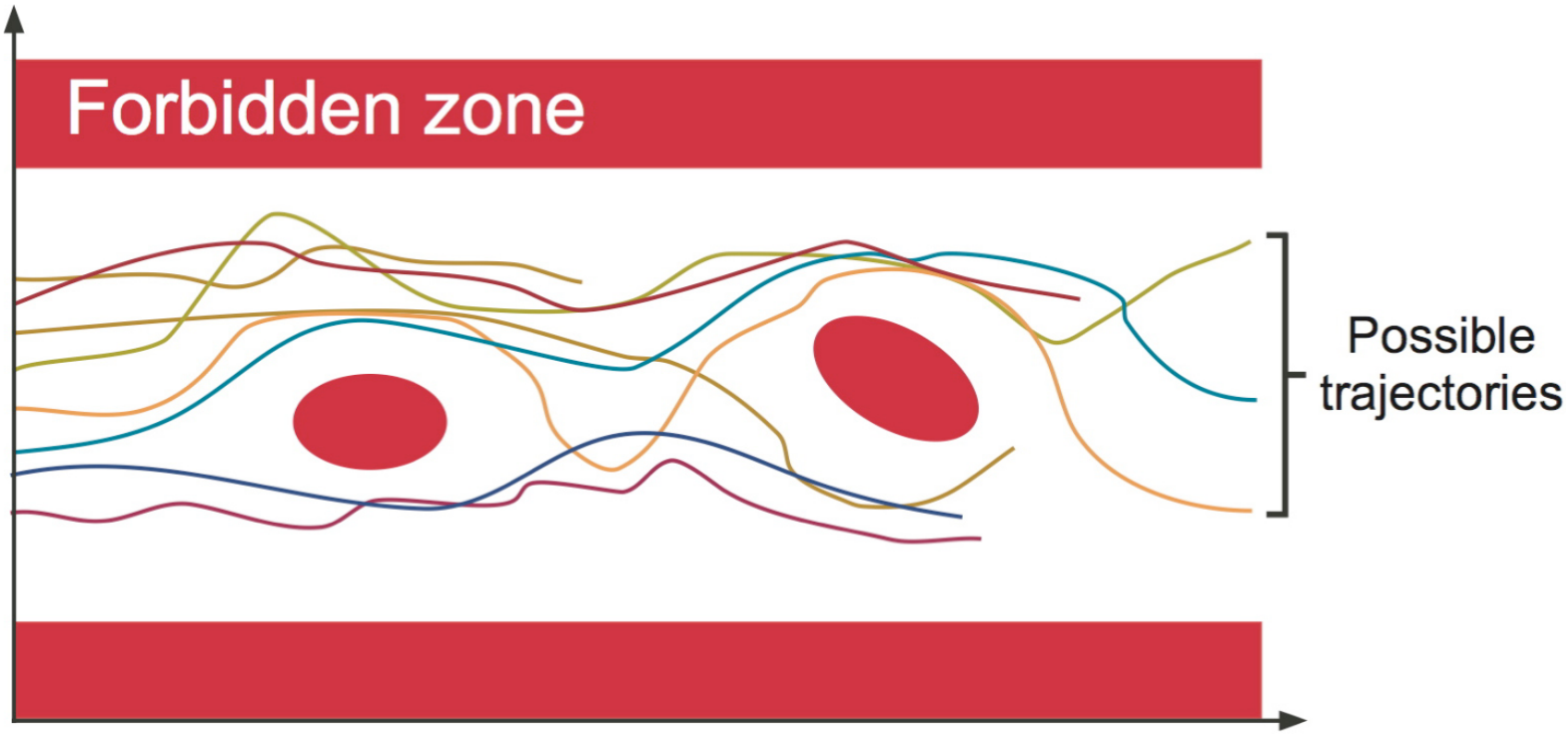
- Often impossible to compute the set of reachable states precisely
- Lets compute them on some level of abstraction

Traces of program

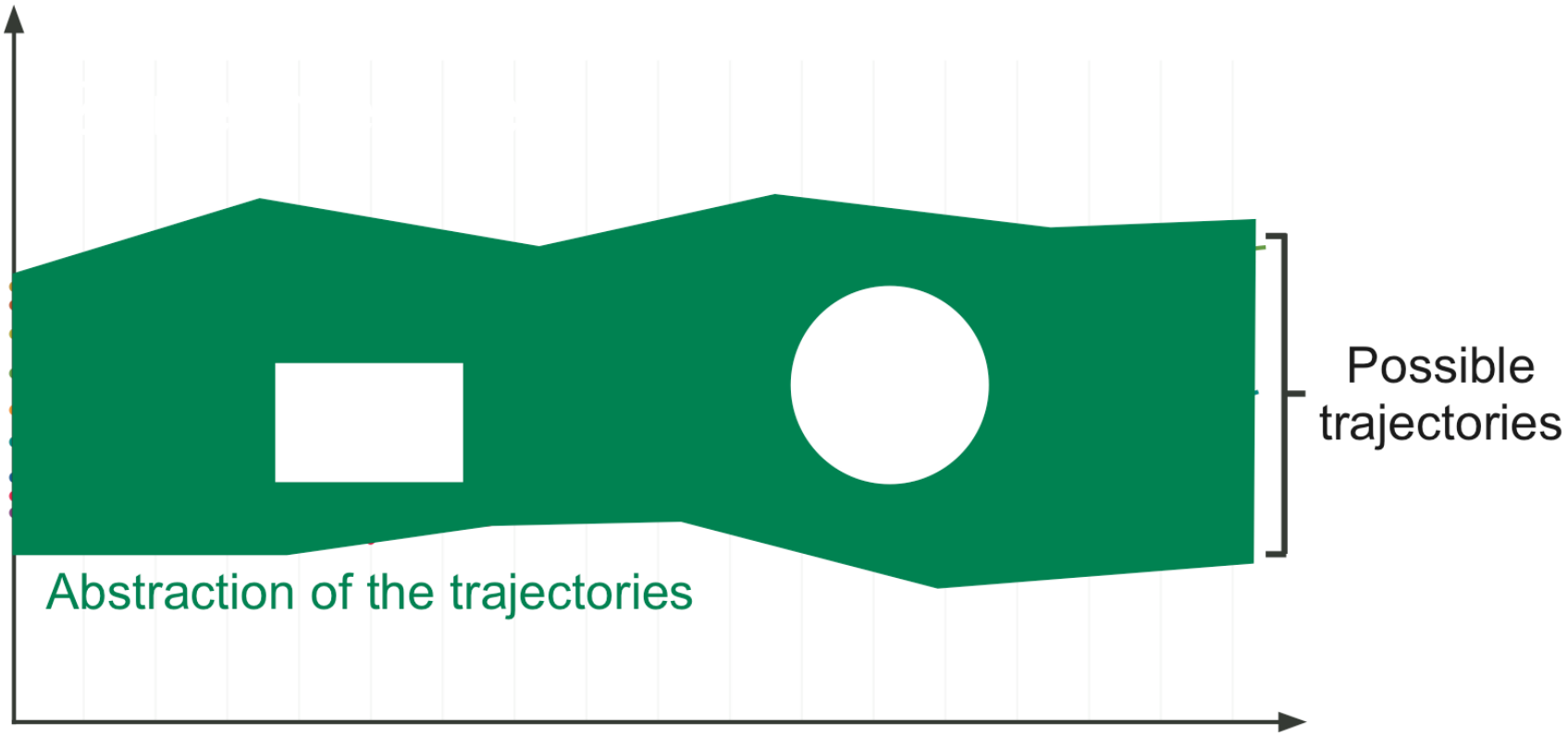


- Evolution of program state through time

Error states



Abstraction of trajectories



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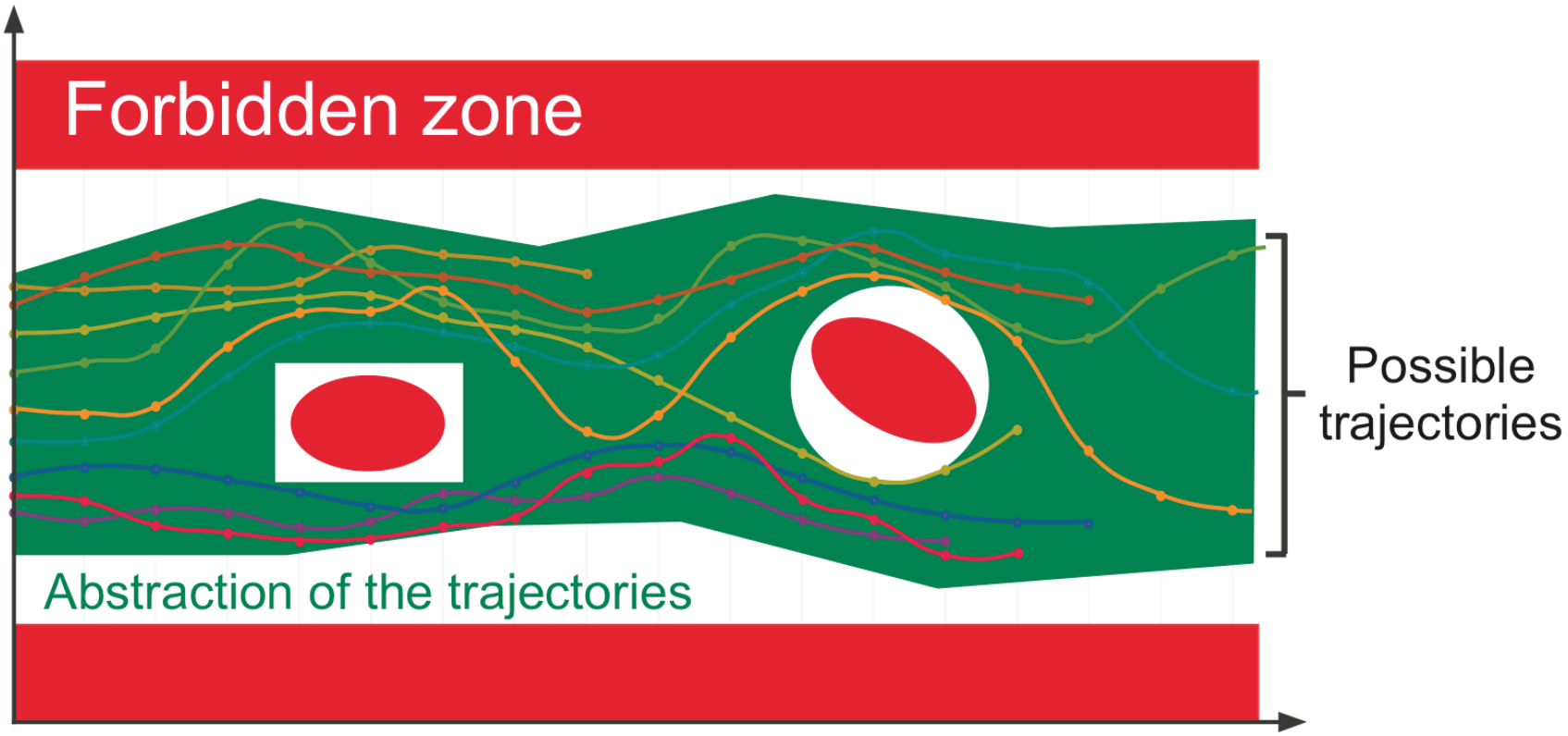
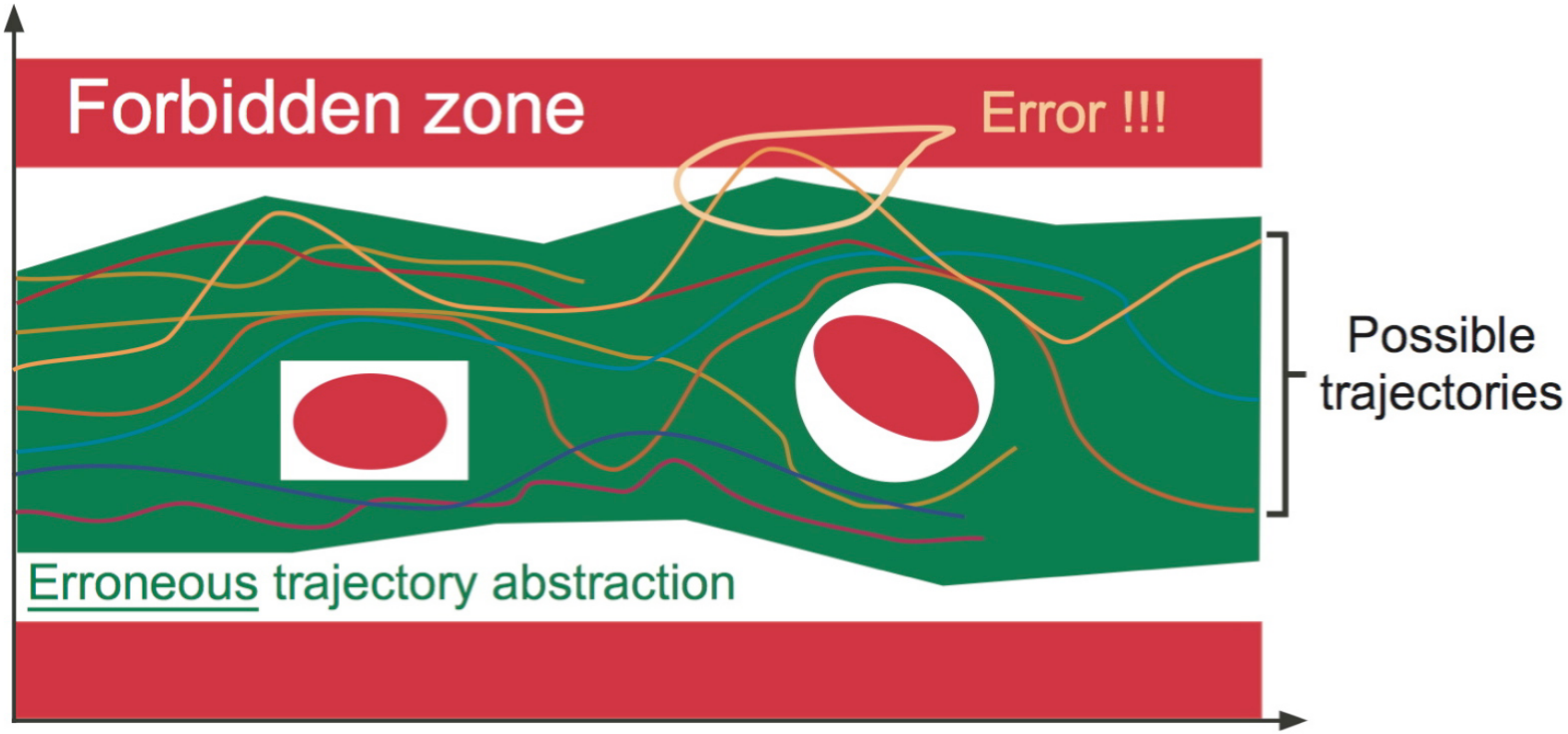


Soundness of abstract interpretation



- We want to exclude unsound abstractions

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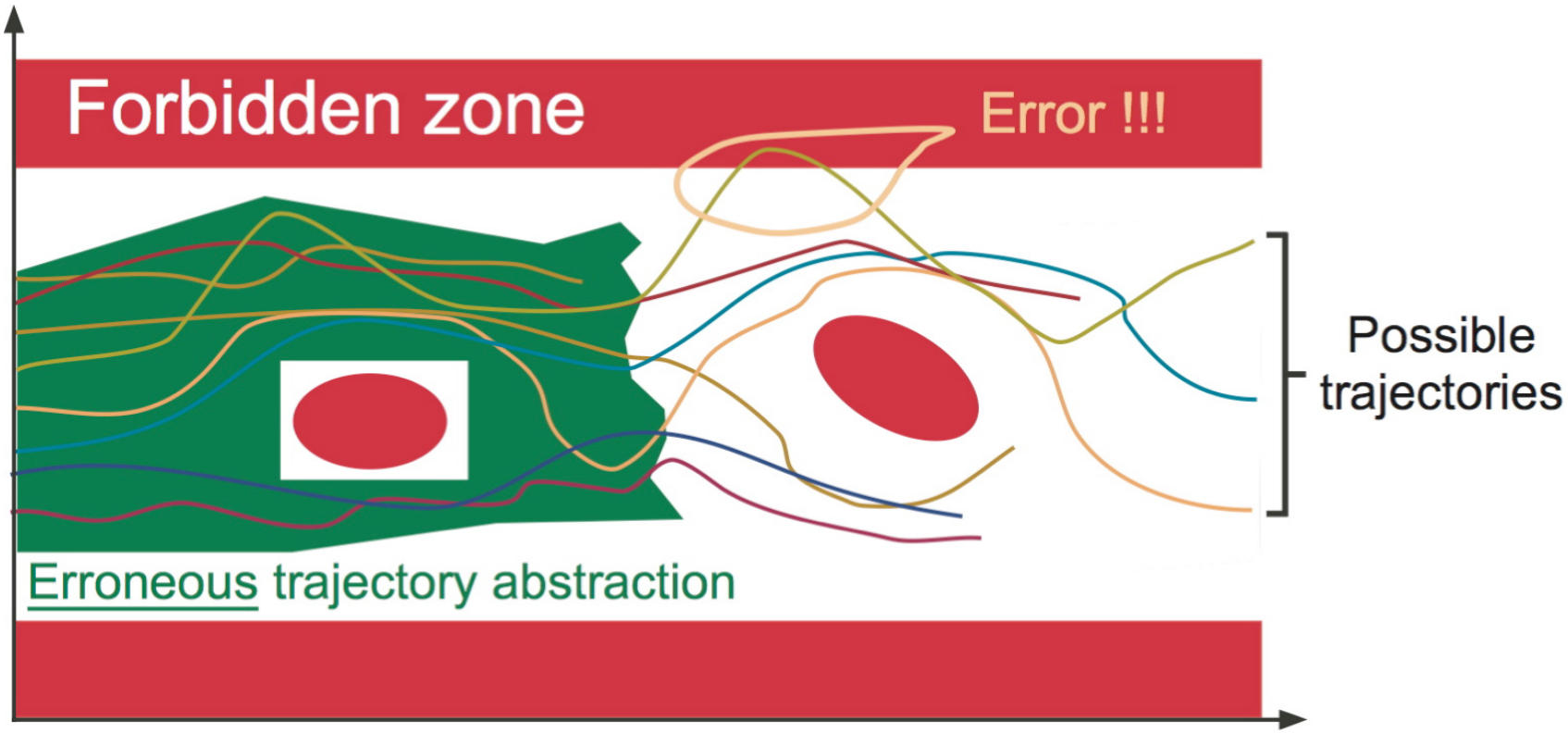
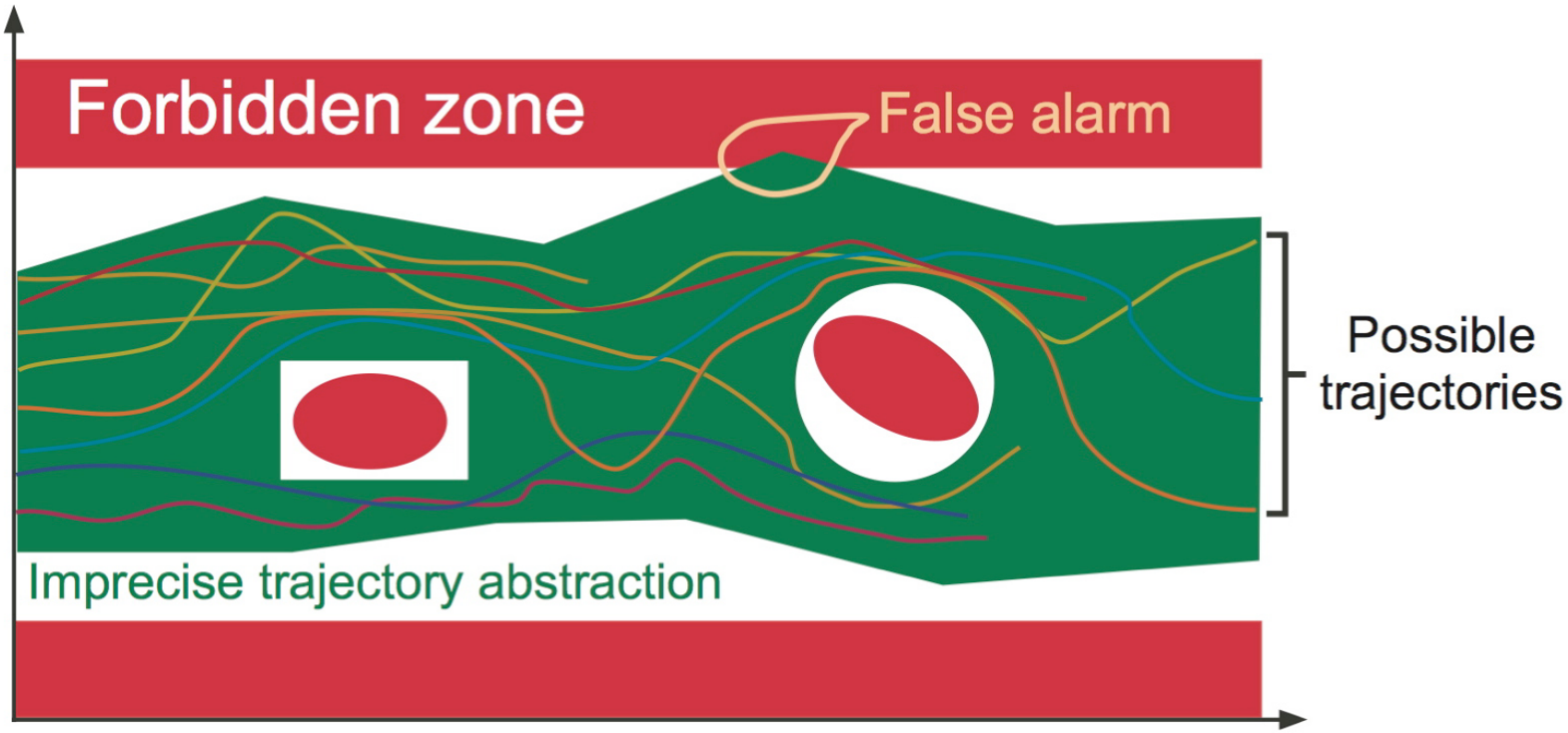


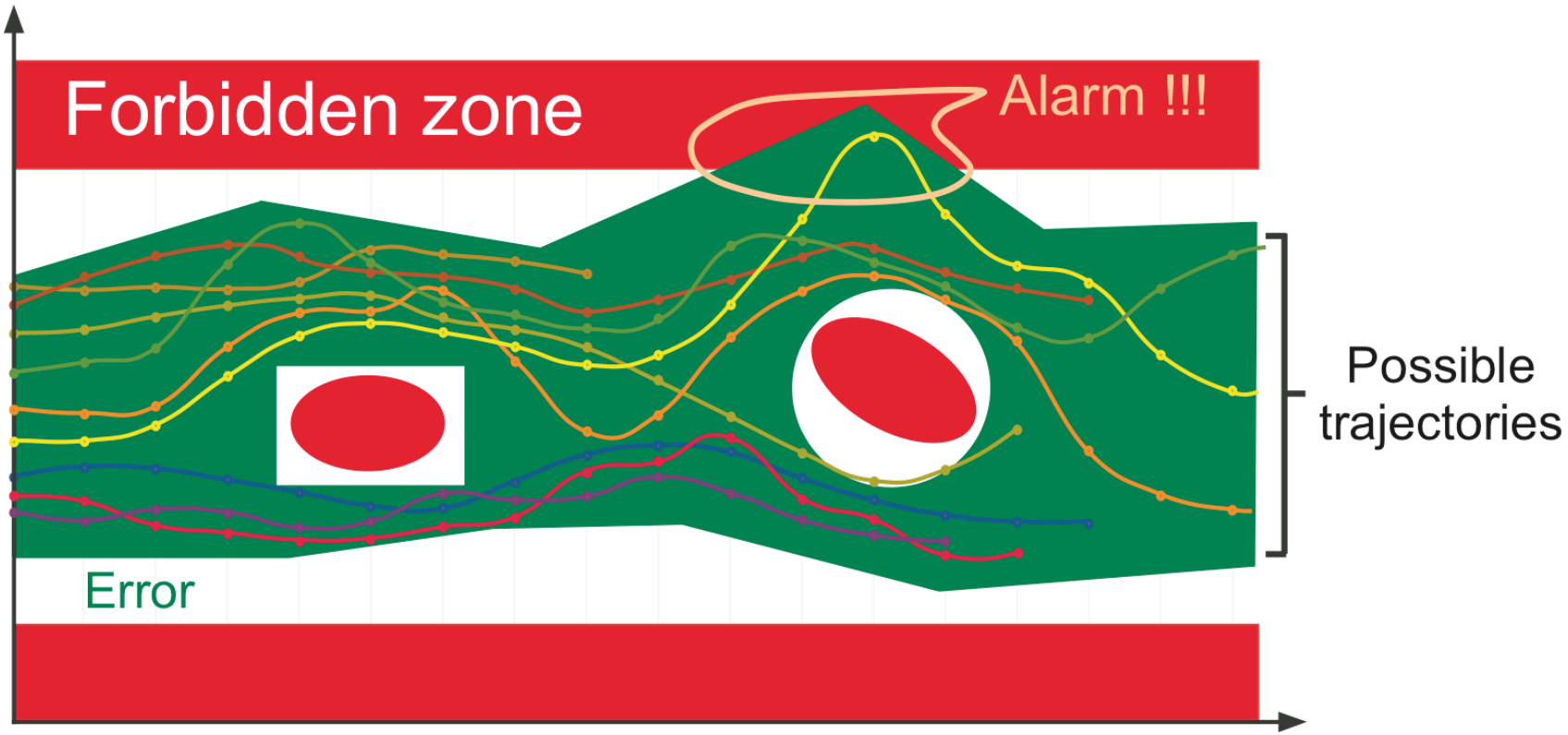
- Bounded model-checking

Over-approximation



- False positive

Good abstraction



Preliminaries

1. Partially ordered sets and lattices
2. Monotone functions
3. Fixpoint computation
4. Strongest postcondition

Abstract intersections and unions

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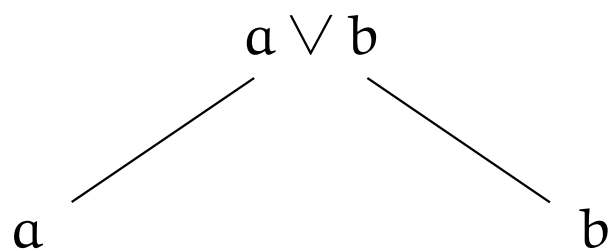

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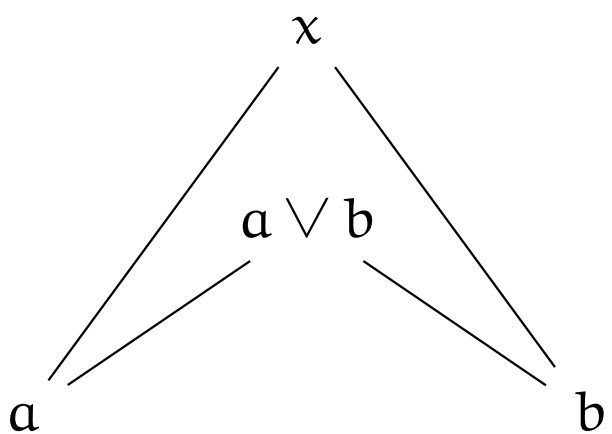

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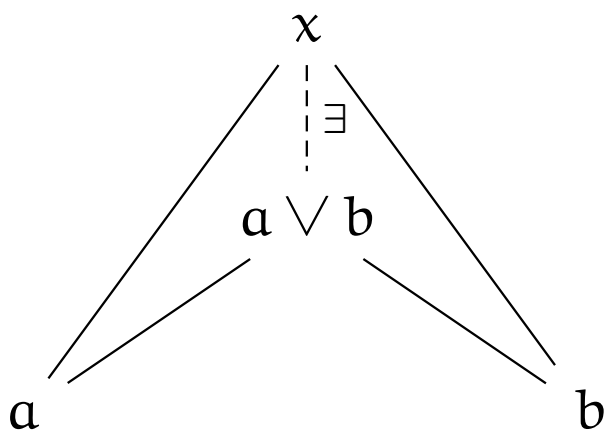

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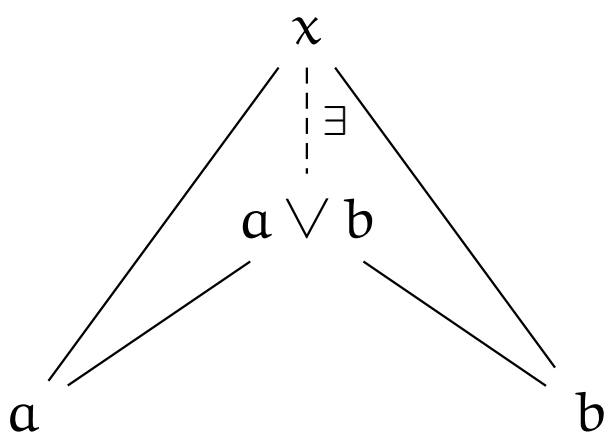

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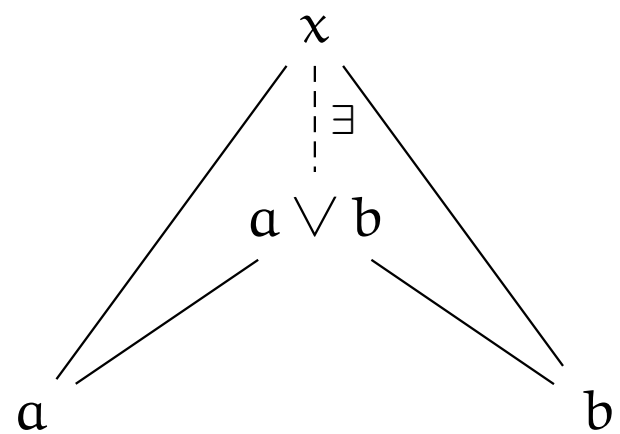

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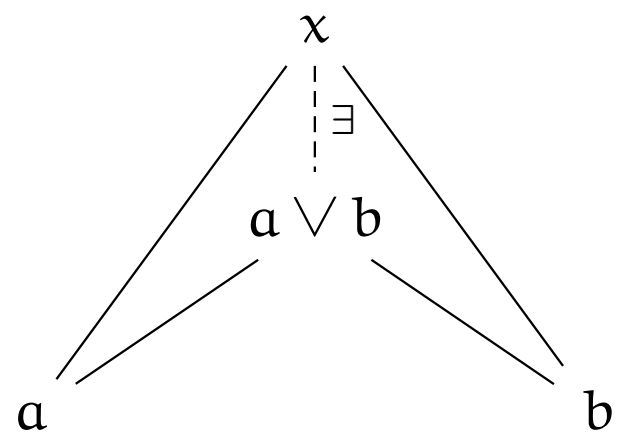

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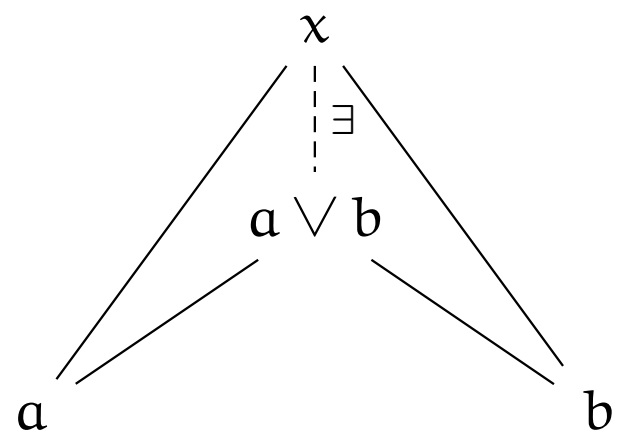

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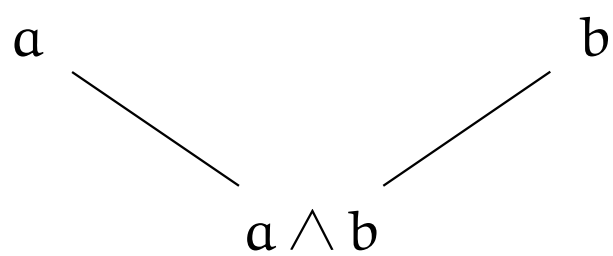

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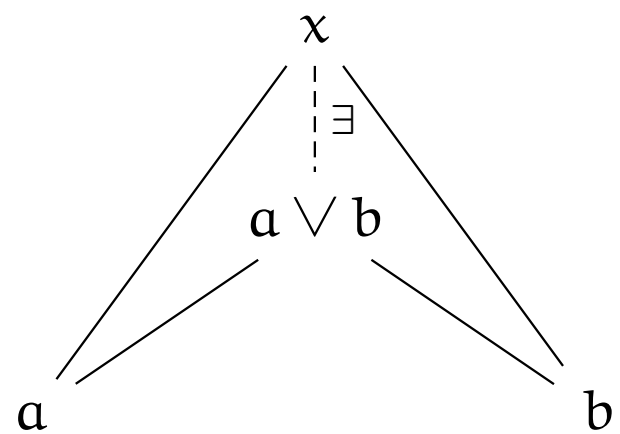

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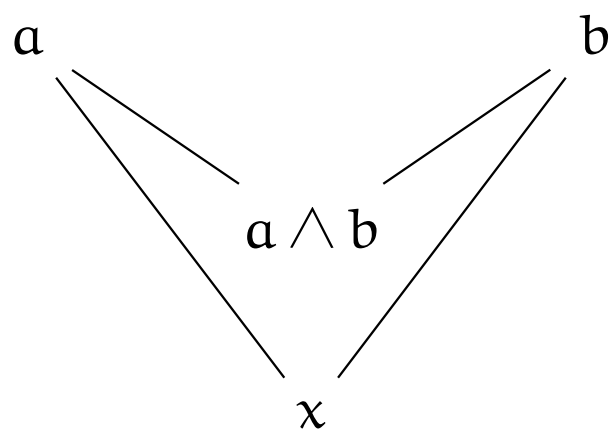

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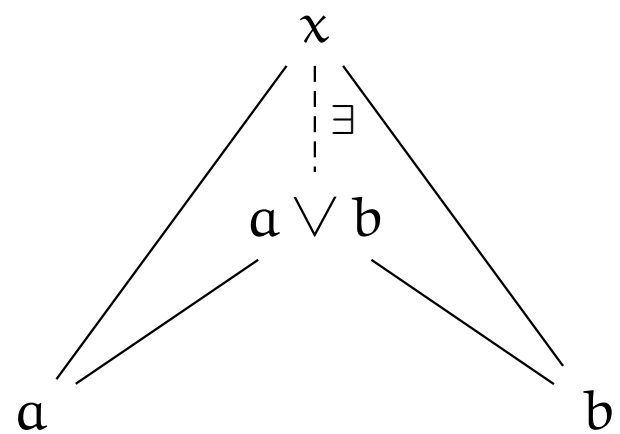

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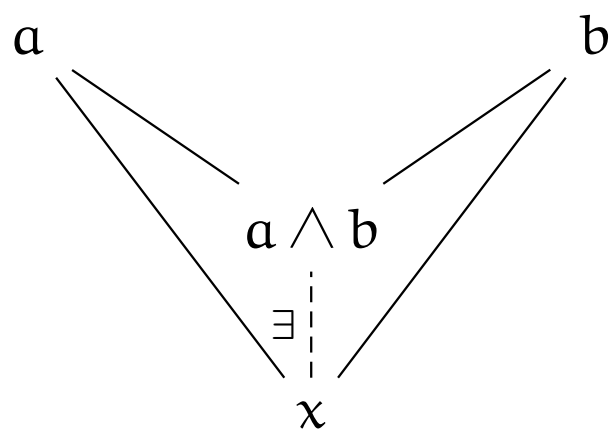

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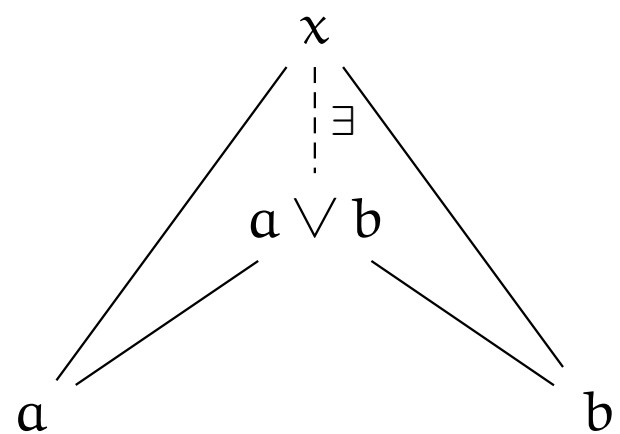

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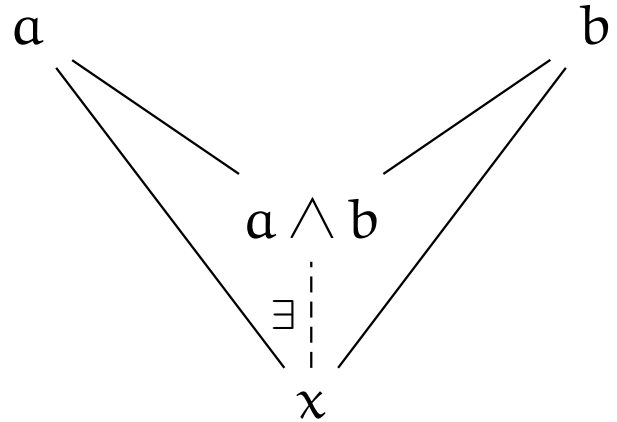

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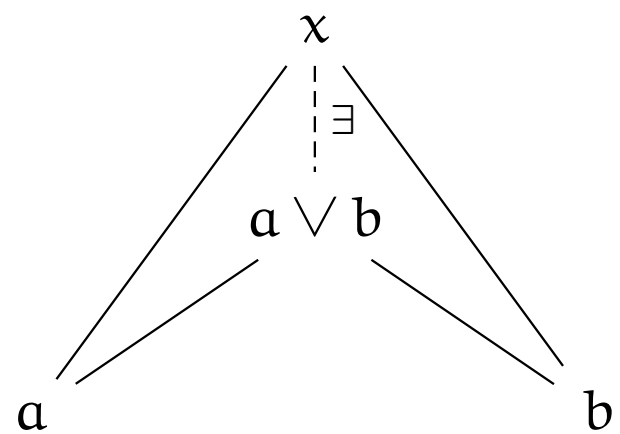

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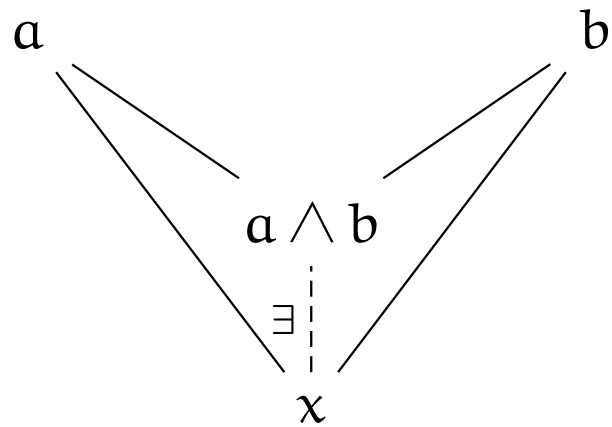

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- $(\mathrm{Unit} \cup \{\emptyset\}, \subseteq)$ for a set Unit of unit intervals (i.e. $[a, a+1)$)

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A complete lattice always has

- the greatest element ($\top$) called top
- the least element ($\bot$) called bottom

Monotonic function

Let $(P, \leqslant), (R, \sqsubseteq)$ are posets

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A monotone function $f : P \to P$ on a poset $(P, \leqslant)$ is called a transformer.

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- $\lambda x.(x + 1)$ on $(\mathbb{Z}, \leqslant)$
- $\lambda[x, y].([x + 1, y + 1])$ on Int_L
- $\lambda x.(-x)$ on $\mathrm{Sig}_L = (\{-1, 0, 1, \top, \bot\}, \leqslant)$

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What is gfp and lfp in $\mathrm{Sig}_{\mathrm{L}}$?

Theorem (Kleene)

Let $(\mathrm{L}, \leqslant)$ *be a complete lattice of finite height and transformer* f. *Then there exists* $\mathrm{n} \in \mathbb{N}$ *such that for all* $\mathrm{k} \in \mathbb{N}$ *it is* $\mathrm{f}^{\mathrm{n}}(\bot) = \mathrm{f}^{\mathrm{n}+\mathrm{k}}(\bot)$ *and* $\mathrm{f}^{\mathrm{n}}(\bot) = \mathrm{lfp}(\mathrm{f})$.

Fixpoint computation

Algorithm

```
1    x := ⊥;
2    do {
3       t := x;
4       x := f(x);
5    } while (x ≠ t);
```

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Given expression e and programs state s, then $sp(s, e)$ is a strongest postcondition

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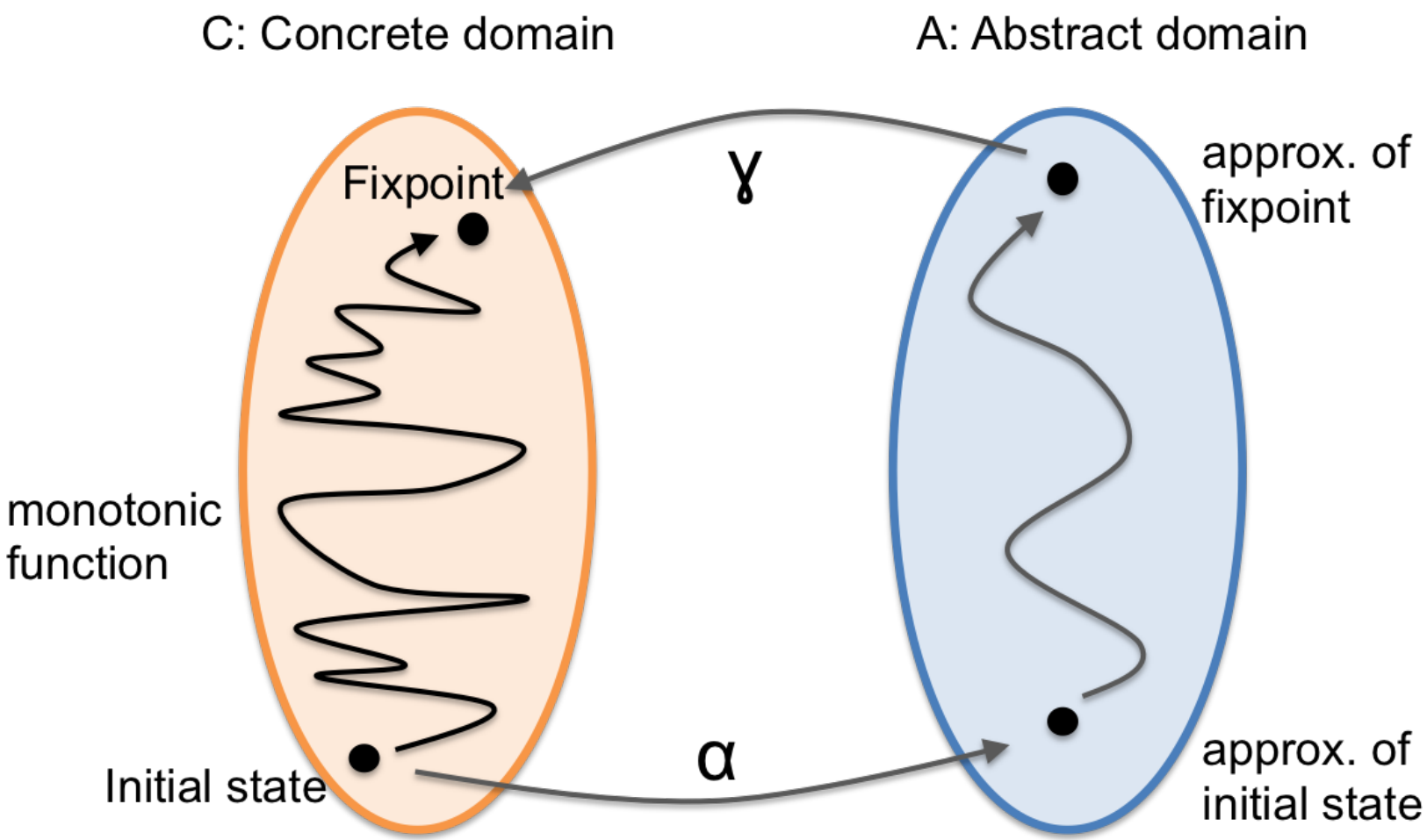
Concrete and abstract domains



Concrete and abstract domains

Except meet and join operations we can equip lattice by other transformers.

Concrete domain – $(C, \leqslant, \wedge, \vee, \{f_1, \ldots, f_k\})$
where $f_1, \ldots, f_k$ are concrete transformers

Abstract domain – $(A, \sqsubseteq, \sqcap, \sqcup, \{af_1, \ldots, af_k\})$
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Idea of abstract interpretation

Consider the assignment: `c = a + b`

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Interpreter (concrete domain):

$$\begin{bmatrix} \texttt{a:10} \\ \texttt{b:-1} \\ \texttt{c:3} \end{bmatrix} \xrightarrow{\texttt{c = a + b}} \begin{bmatrix} \texttt{a:10} \\ \texttt{b:-1} \\ \texttt{c:9} \end{bmatrix}$$

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Each abstract state represents set of concrete states.

Abstract Interpretation Recipe: Setup

Given control-flow graph (V, E, r), where

- $V = \{v_1, \ldots, v_n\}$ is set of program locations
- $E \subseteq V \times V$ are program transitions
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Example: $a = 0$, $e =$ `x := x + 1`

Running Abstract Interpretation

Compute lfp for each program state, where sp_A computes one iteration of interpretation in abstract domain.

```
1        // a
2        x := 0;
3        // b
4        while (x ⩽ 10) {
5          // d
6          if (x > 1)
7               // e
8               x := x + 3
9          else
10              // f
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13       }
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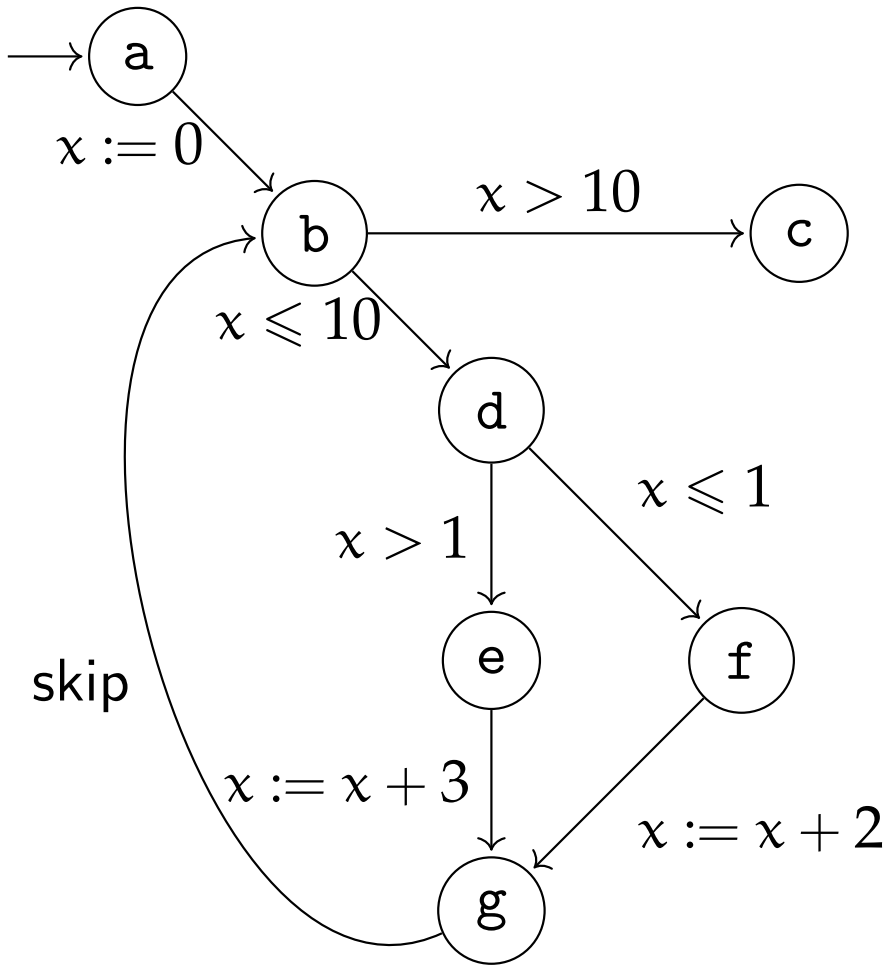
```



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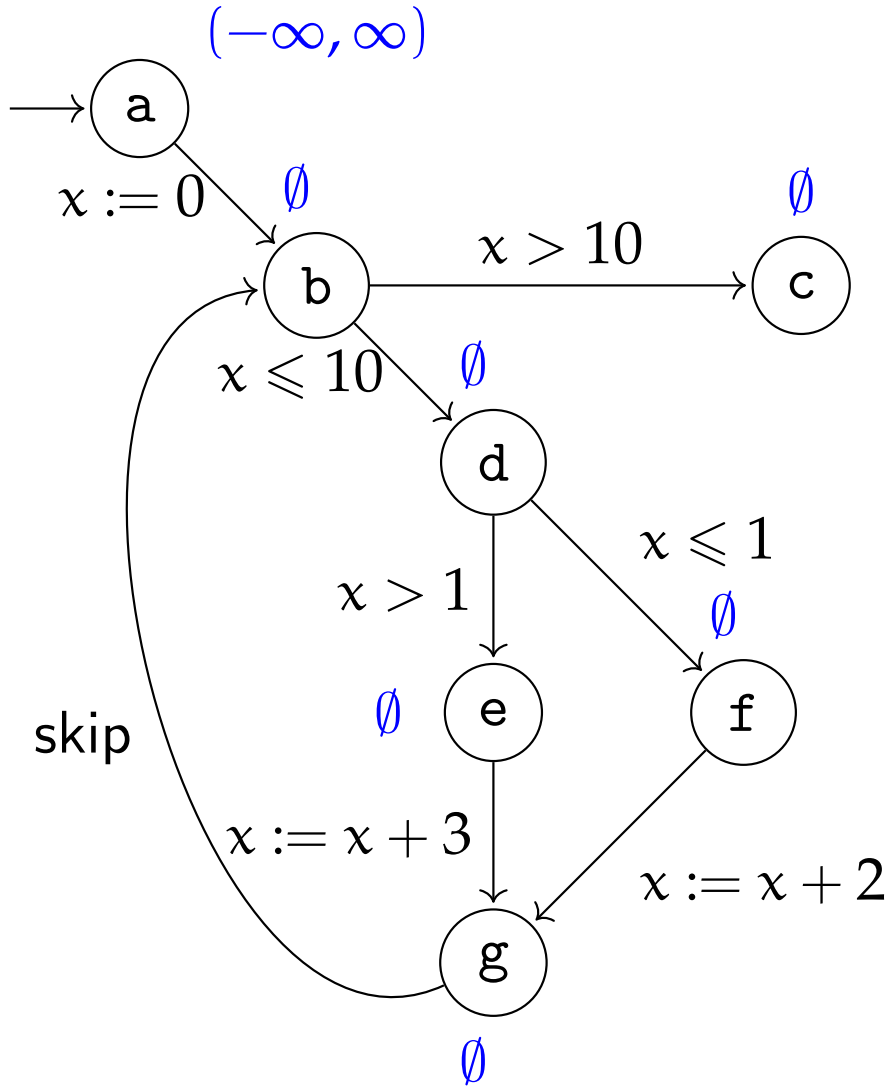

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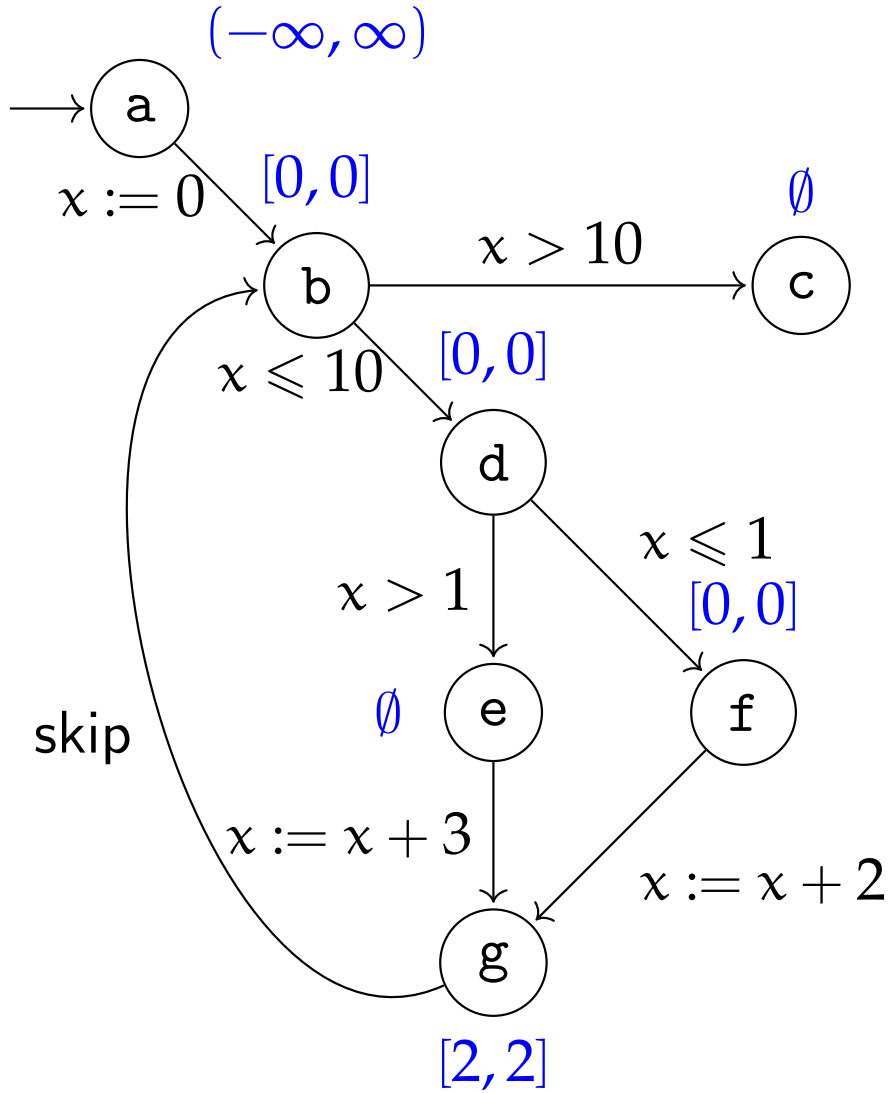

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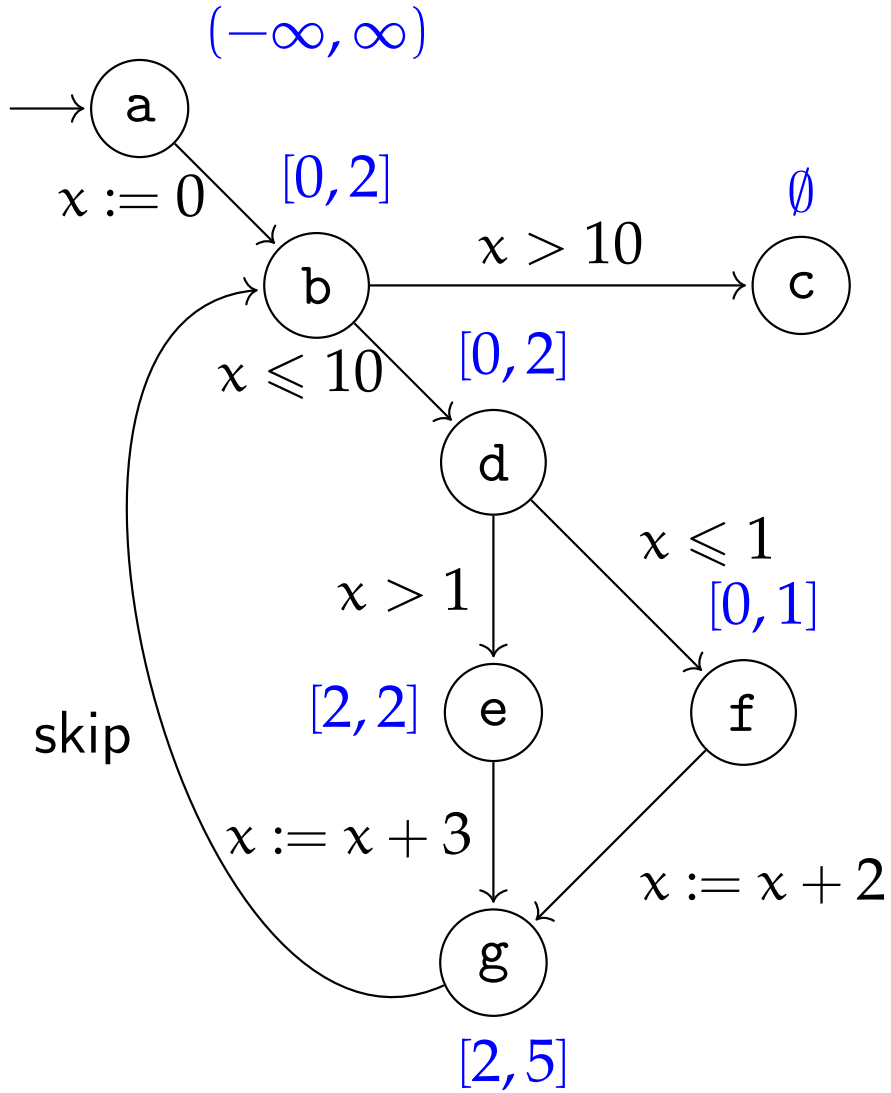

Running Abstract Interpretation

The resulting fixpoint describes an inductive program invariant.

```
1        // a
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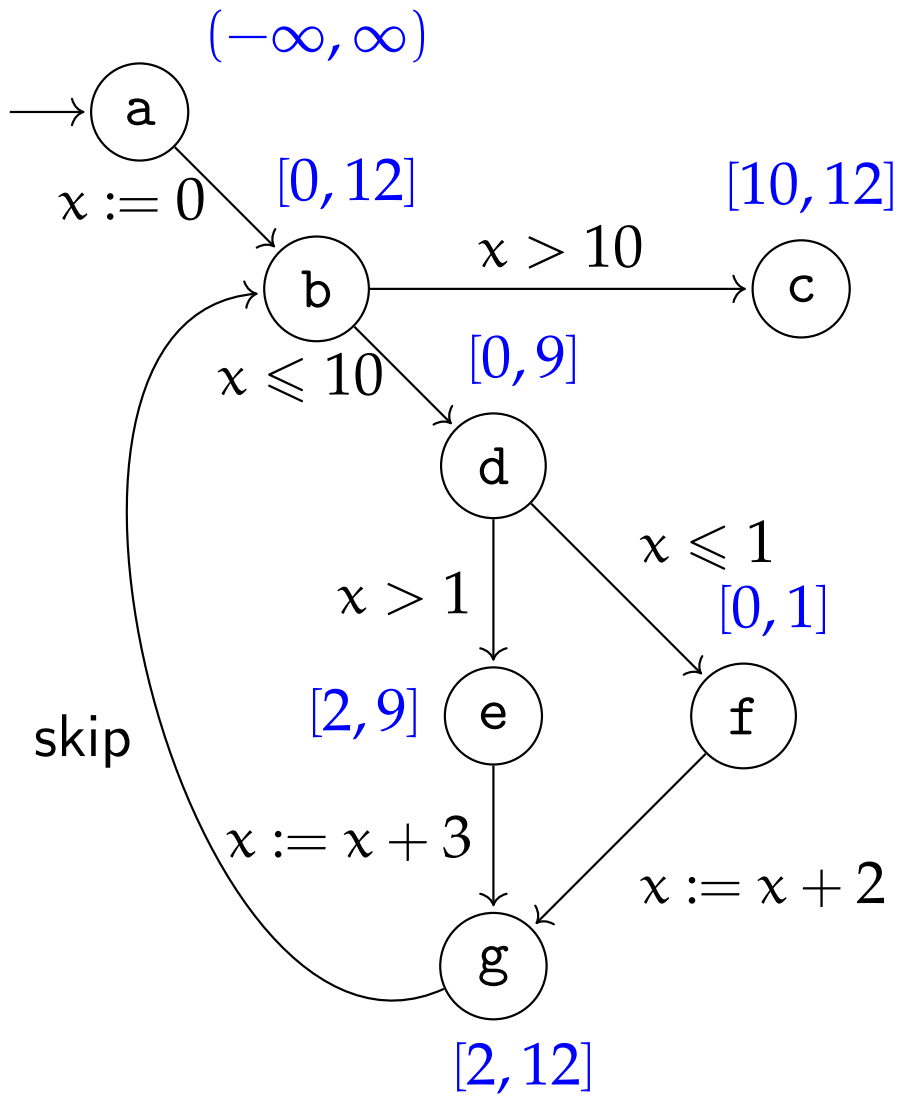
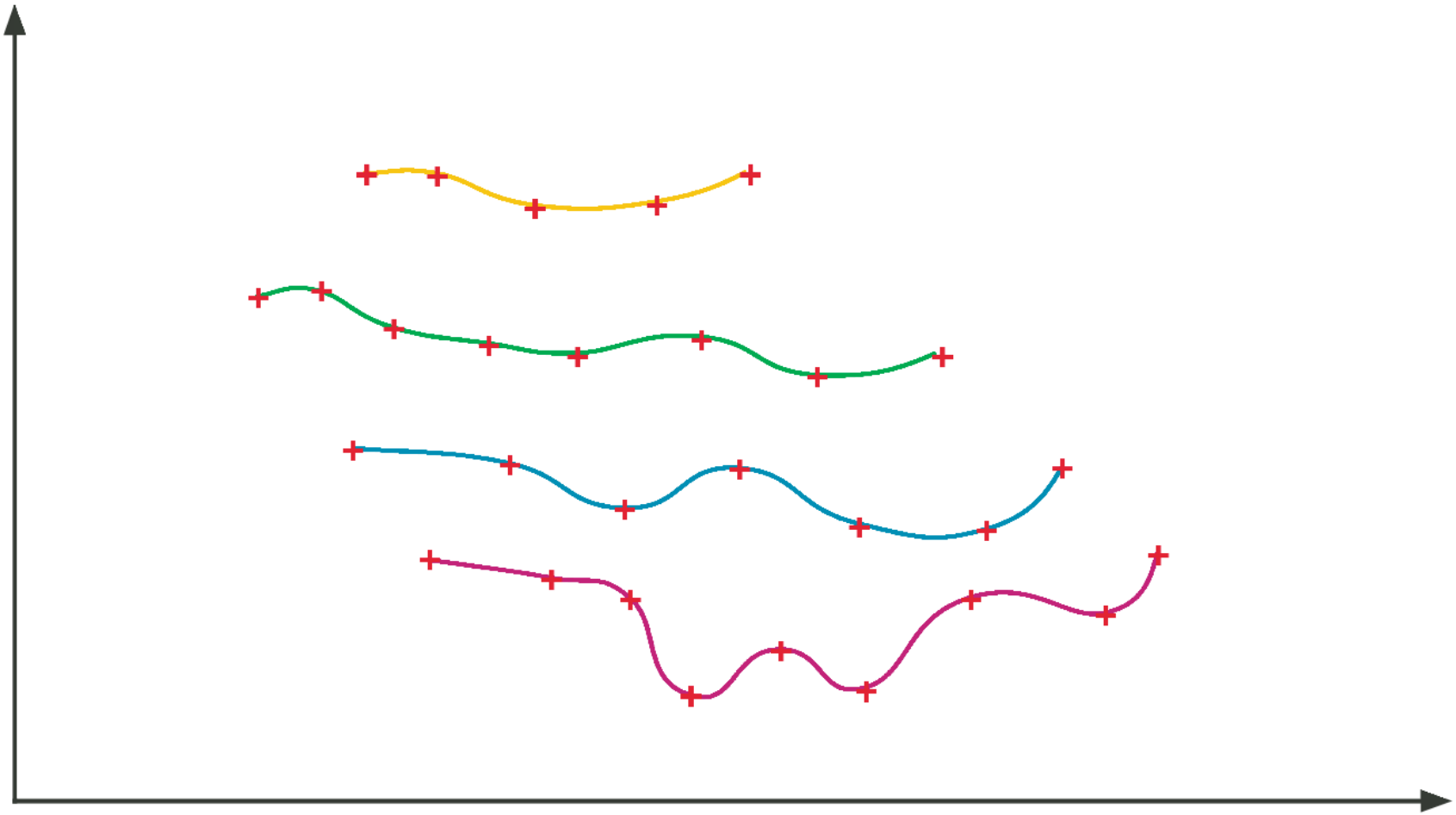
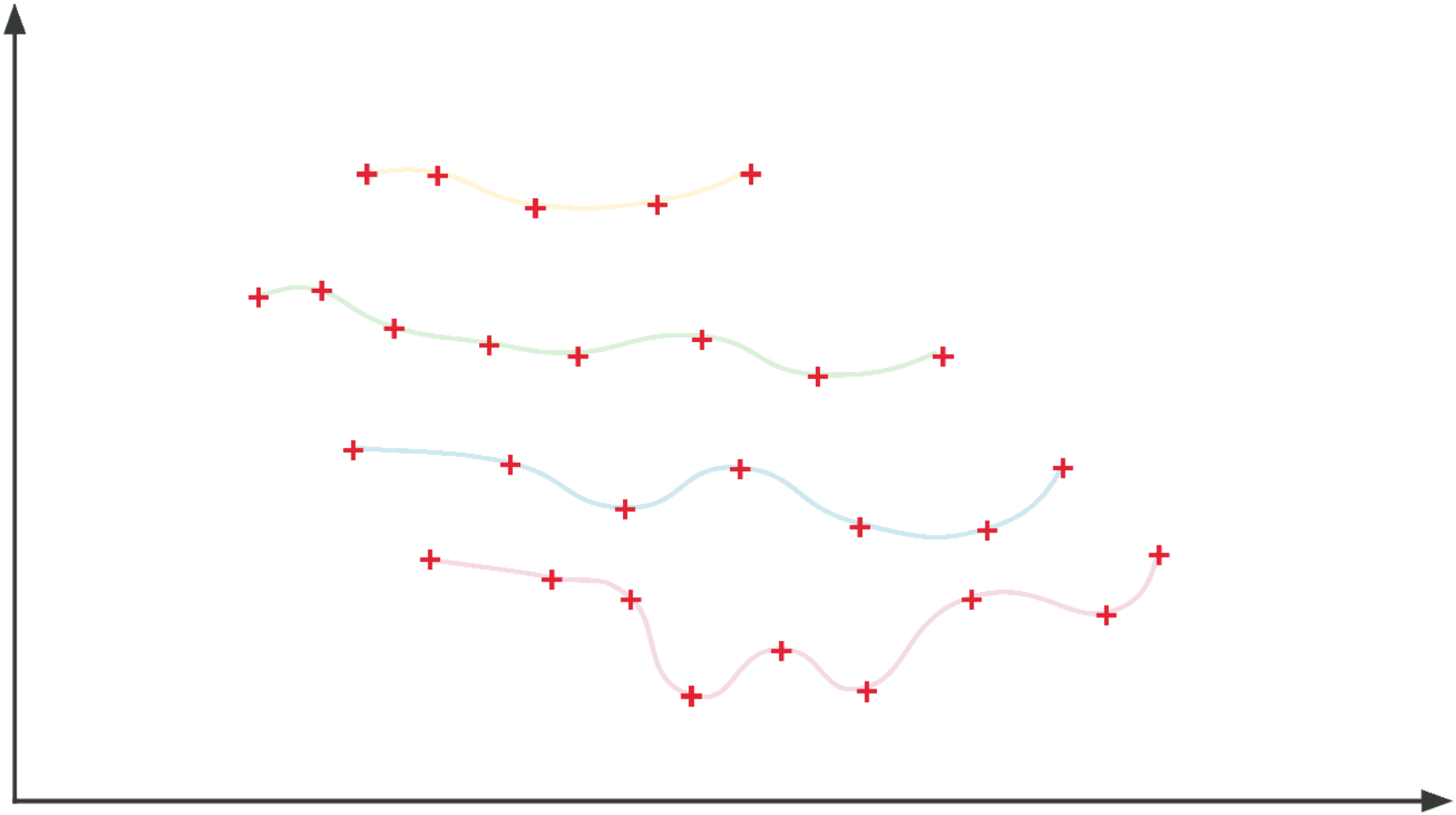


Abstractions



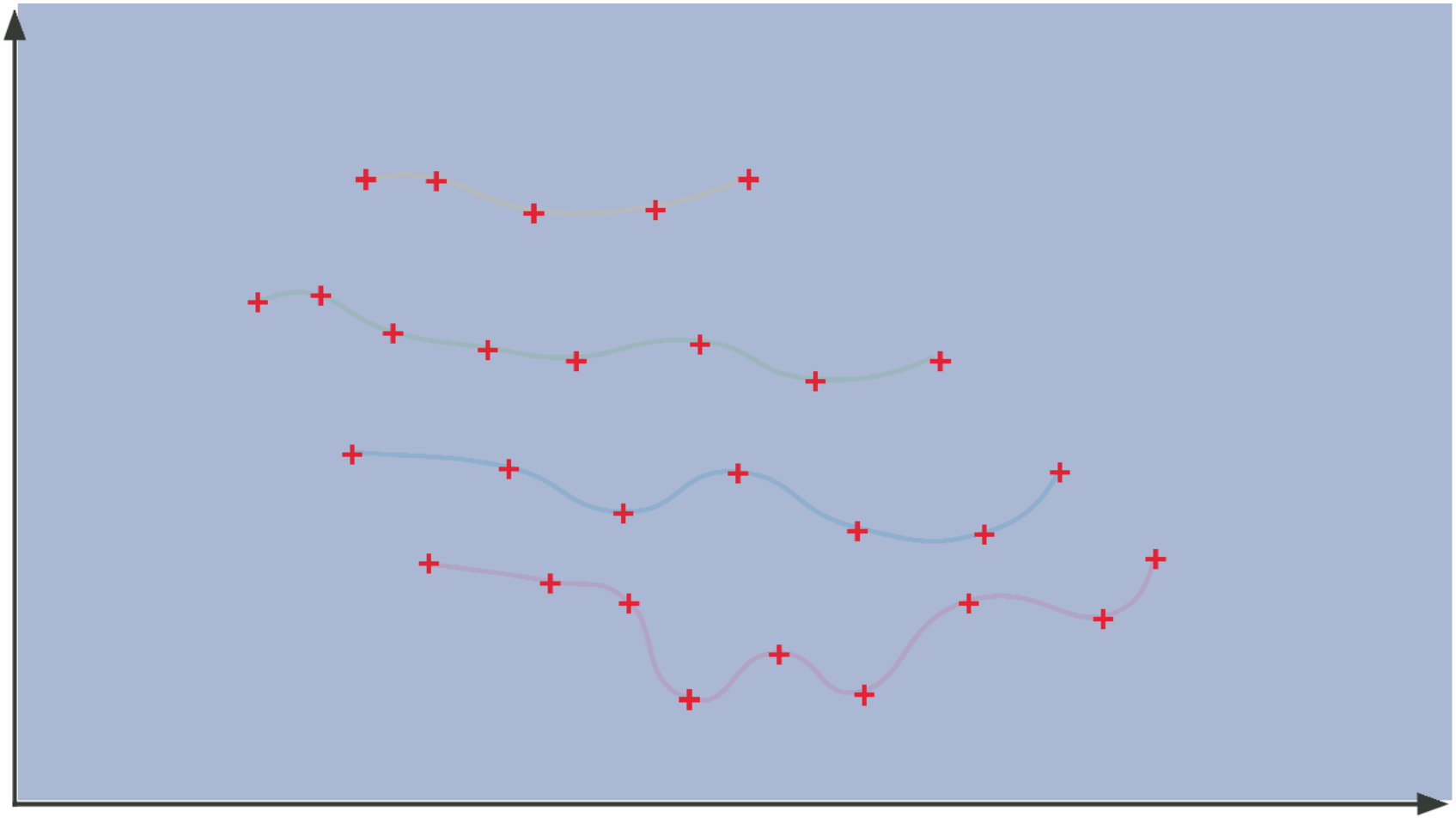
- We want to abstract this set of traces with two variables x and y

Abstractions



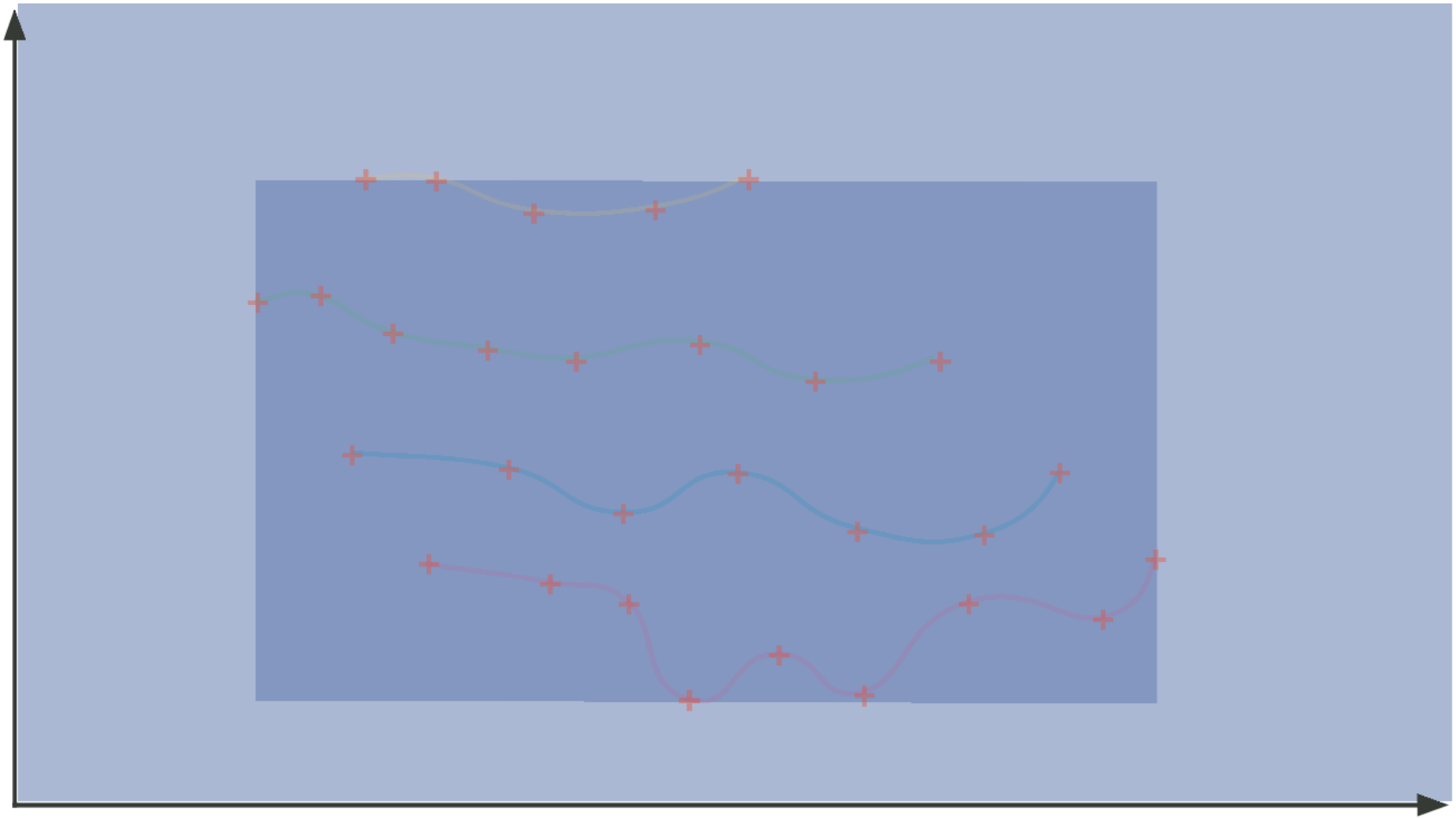
- Decomposition into set of local invariants on memory states attached to each program point

Abstractions: Sign analysis



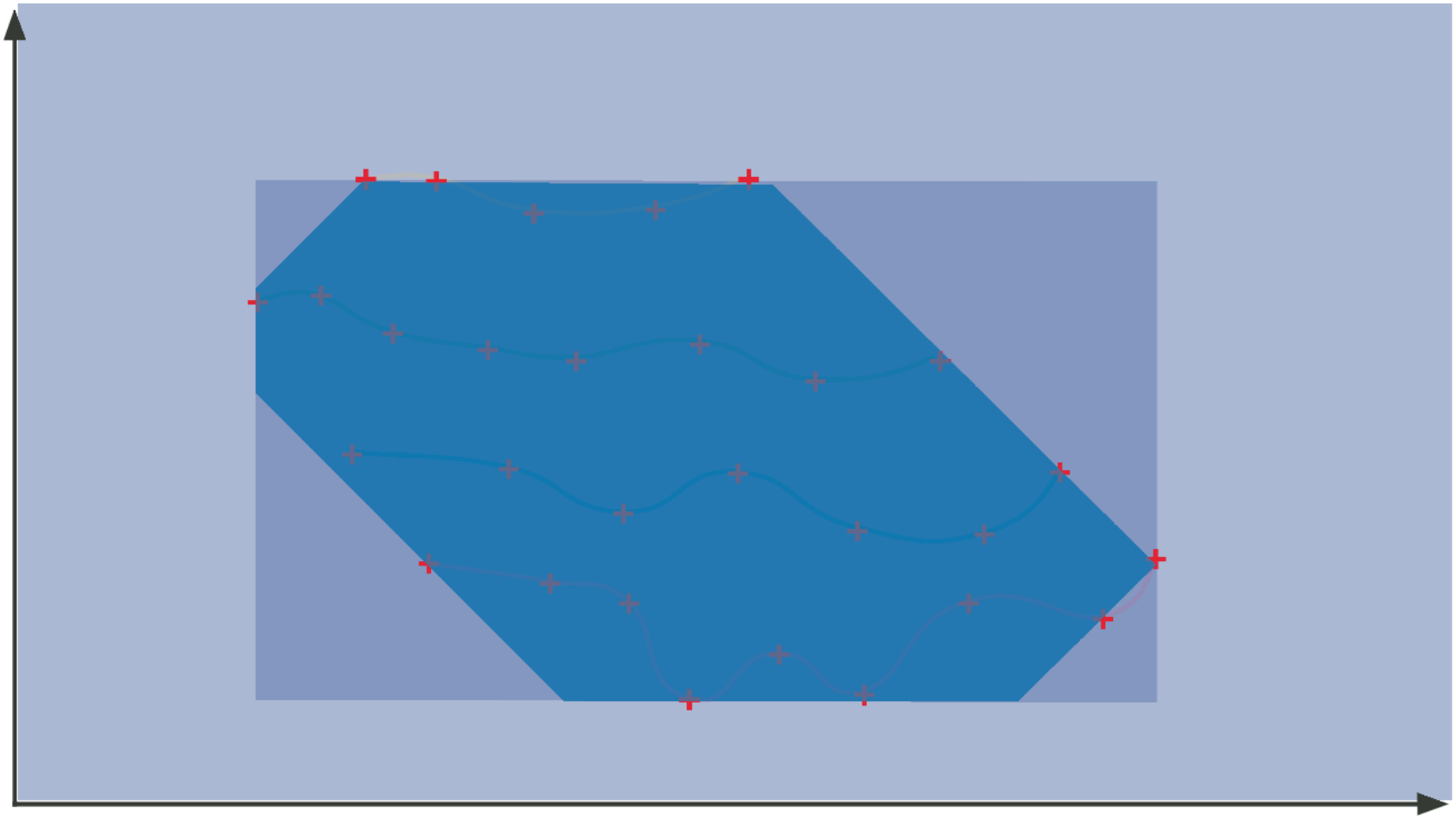
- Local invariants: $x \geqslant 0, y \geqslant 0$

Abstractions: Interval



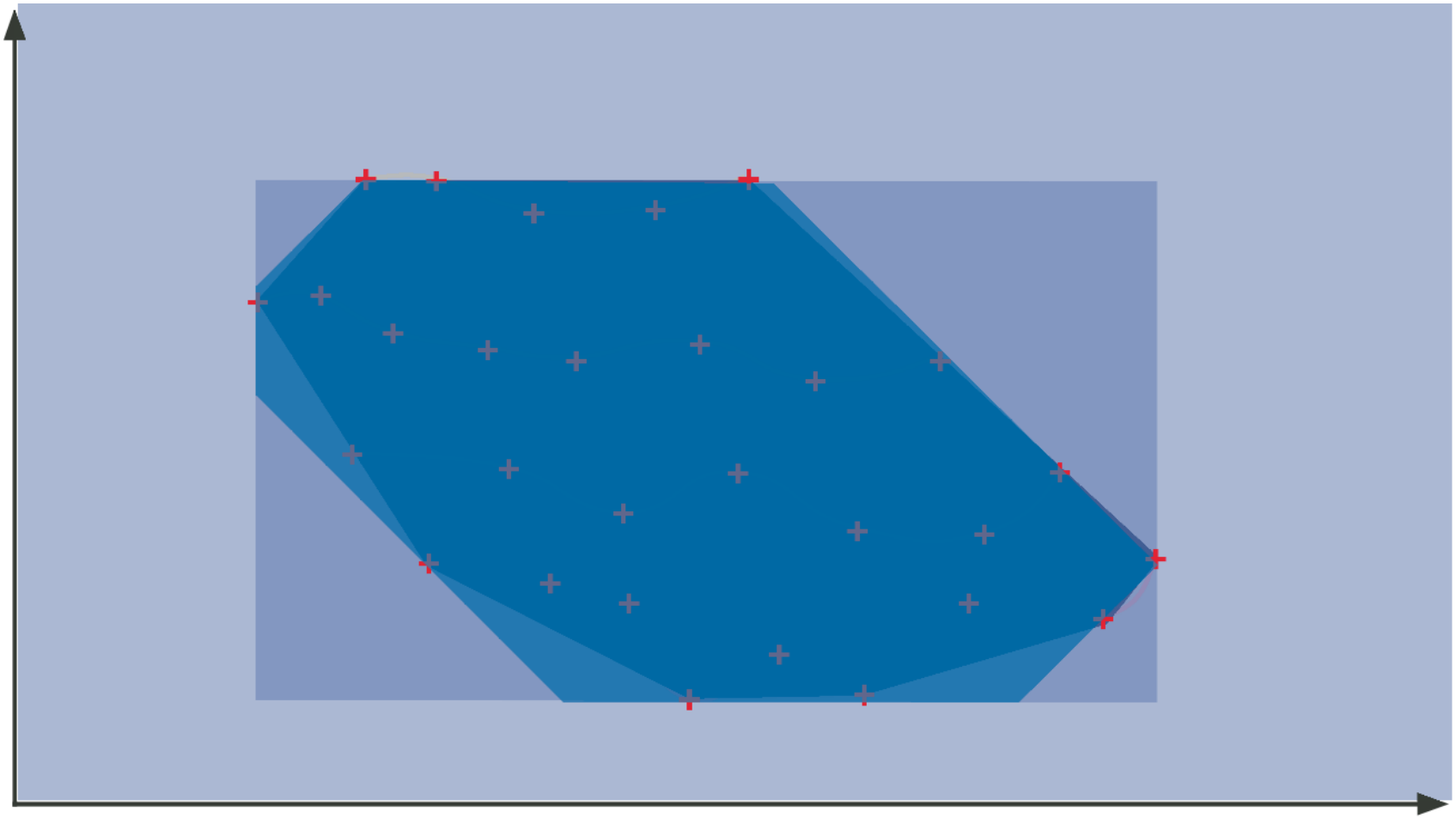
- Local invariants: $a \leqslant x \leqslant b, c \leqslant y \leqslant d$, where a, b, c, d are discovered by analysis
- Cannot prove invariant $0 \leqslant x \leqslant y$

Abstractions: Octagons



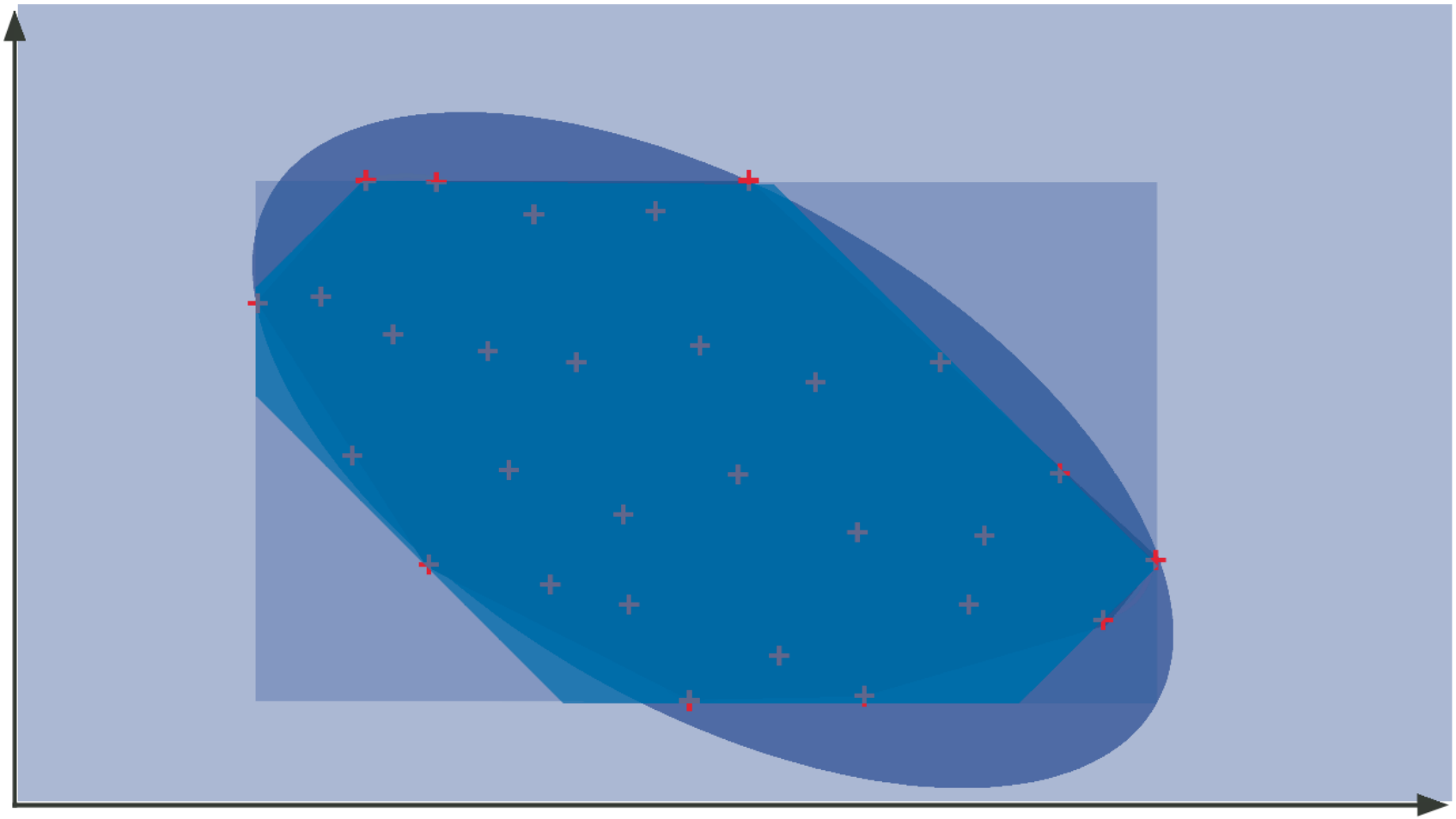
- Local invariants: $x \leqslant a, x - y \leqslant b, x + y \leqslant c$

Abstractions: Convex Polyhedra



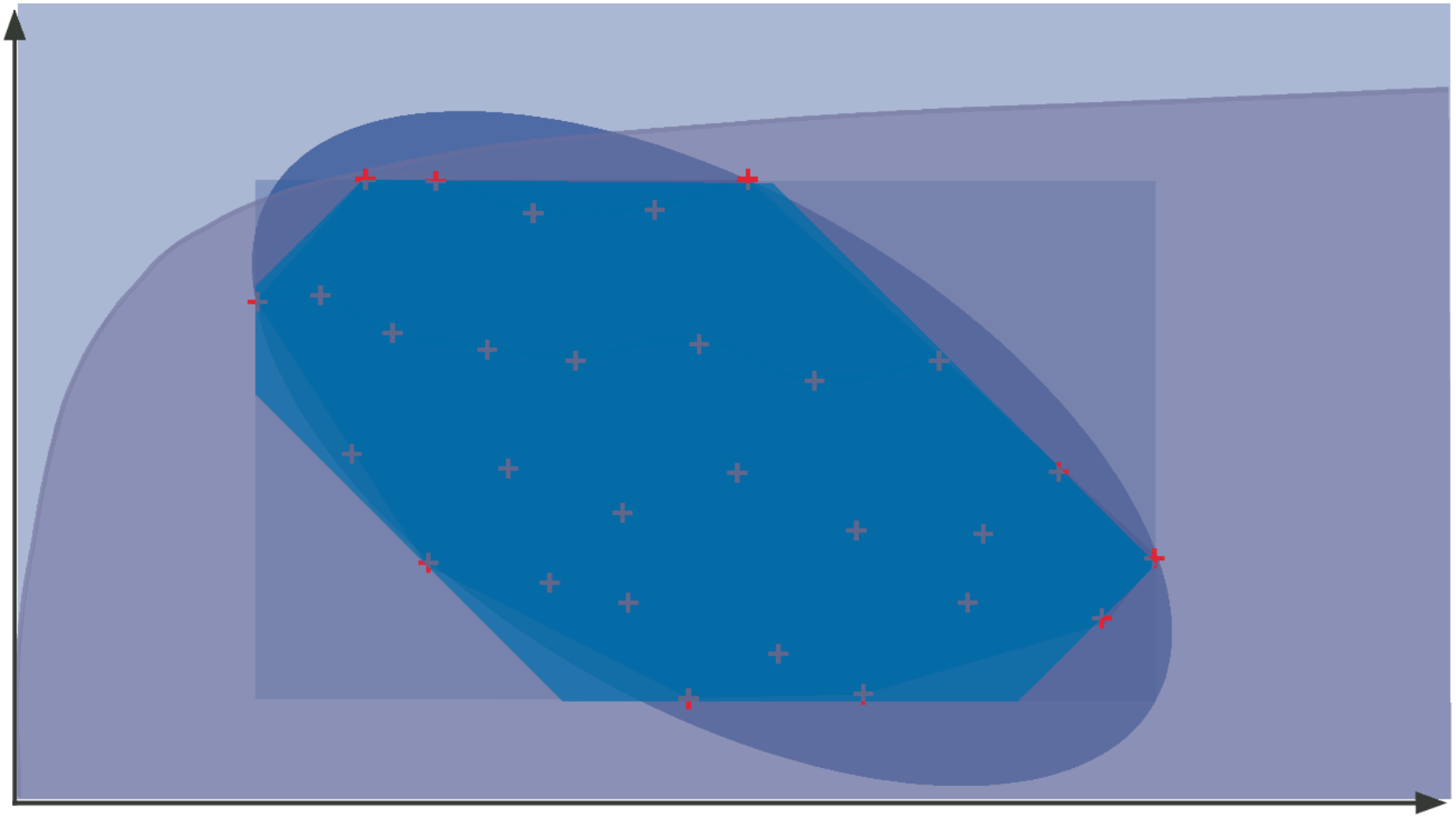
- Local invariants: $a \cdot x + b \cdot y \leqslant c$

Abstractions: Ellipsoids



- Local invariants: $(x - a)^2 + (y - b)^2 \leqslant c$

Abstractions: Exponential



- Local invariants: $a^x \leqslant y$