Acceleration of Abstract Interpretation: Widening and Nowrrowing

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Brno, Czech Republic

9th October 2017

Refresh I

Partially Ordered Set (poset)

- $(P, \sqsubseteq)$, $P \neq \emptyset$
- $\sqsubseteq$ is a binary relation which is reflexive, anti-symmetric, and transitive

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Lattice

- let $(P, \sqsubseteq)$ be a poset
- if $\inf(x, y)$ and $\sup(x, y)$ exists for all $x, y \in P$, then $(P, \sqsubseteq)$ is a *lattice*
- if $\inf(X)$, $\sup(X)$ exists for all $X \subseteq P$, then $(P, \sqsubseteq)$ is a *complete lattice*

Refresh II

Galois Connection

- let $(C, \leqslant)$ and $(A, \sqsubseteq)$ be complete lattices
- functions $\alpha\colon C \to A, \gamma\colon A \to C$ such that
  $\forall c \in C, \forall a \in A : \alpha(c) \sqsubseteq a \Leftrightarrow c \leqslant \gamma(a)$
- also $A \underset{\gamma}{\overset{\alpha}{\leftrightarrows}} C$

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  - where $f^\#\colon A \to A$ approximates f
- if A is finite or has no infinite strictly ascending chains:

$$\mathsf{lfp}(f^\#) = (f^\#)^n(\bot_A)$$

  - for some $n \in \mathbb{N}$, $\bot_A = \inf(A)$
  - can be calculated iteratively

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important abstract domains have strictly ascending chains:

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- we can trade precision for tractability
- accelerate iterative computation
- possibly find a element $a \sqsupseteq \mathsf{lfp}(f^{\#})$

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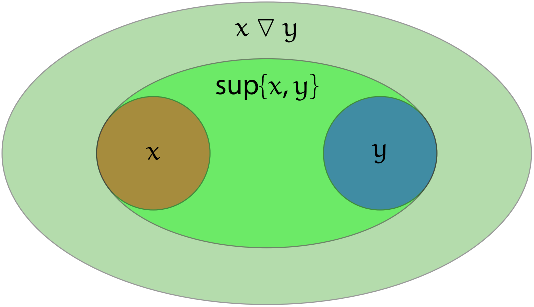

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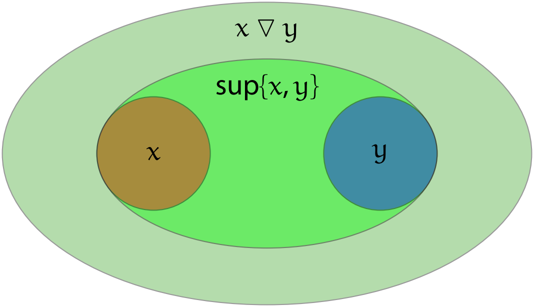

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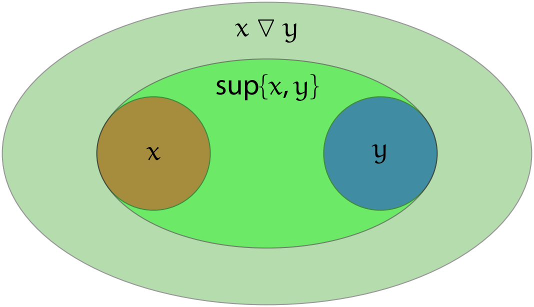

Widening II

- iterative calculation of $\widehat{x} \sqsupseteq \mathsf{lfp}(f^{\#})$:

$$\begin{aligned}\widehat{x}_0 &= \bot \\ \widehat{x}_{i+1} &= \widehat{x}_i \mathbin{\nabla} f^{\#}(\widehat{x}_i)\end{aligned}$$

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- example of $\triangledown$ (intervals):

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  - if the bound is expanding, extrapolate to infinity

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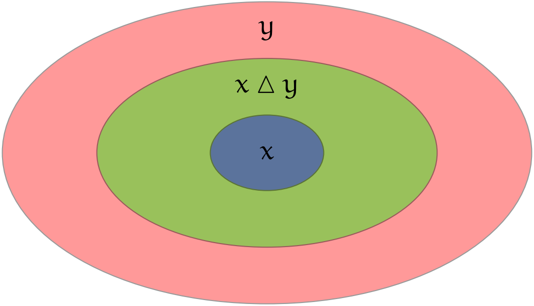

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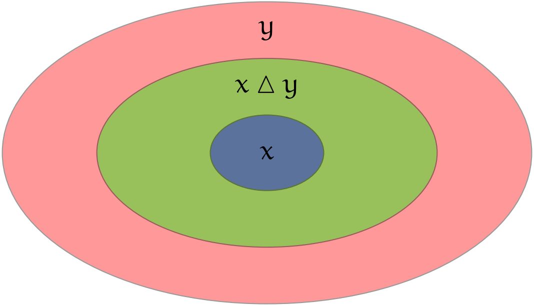

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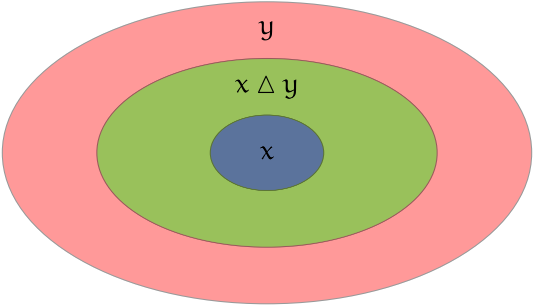

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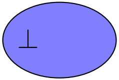
  - prefer finite bounds

Widening and Narrowing in a Diagram



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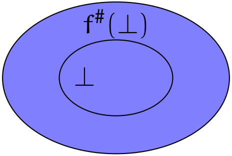

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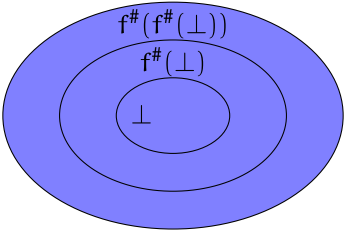

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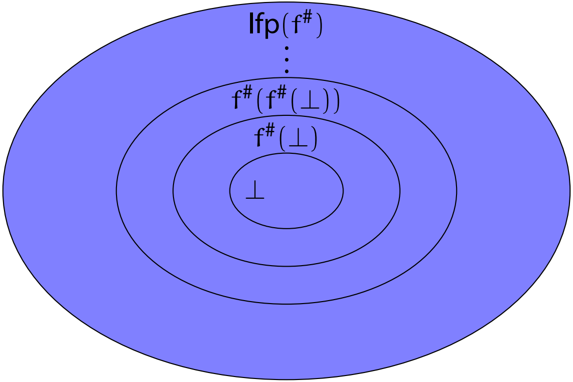

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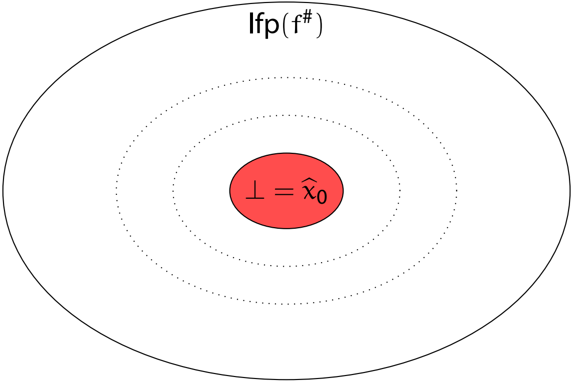

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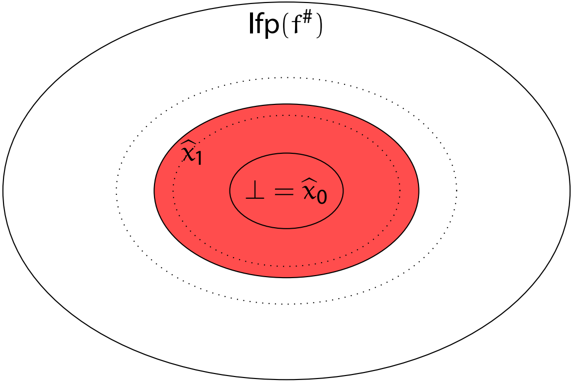

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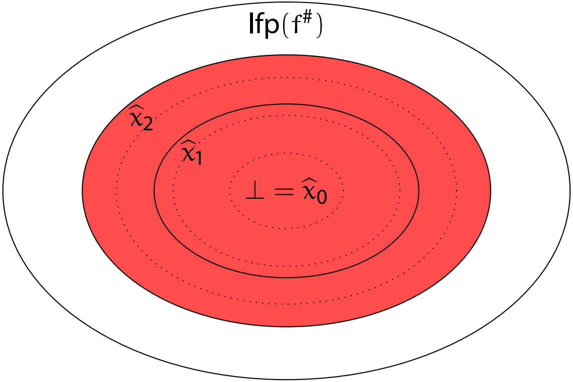

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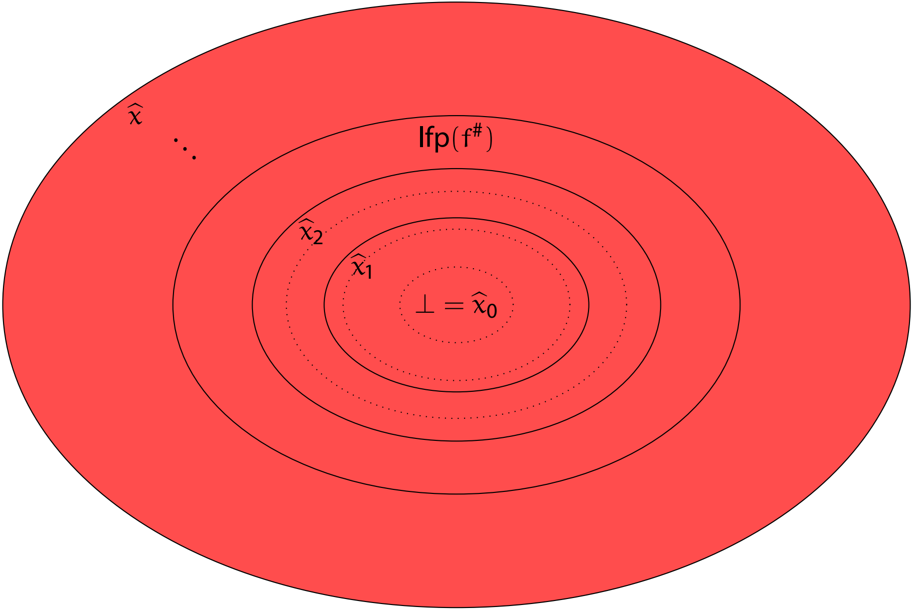

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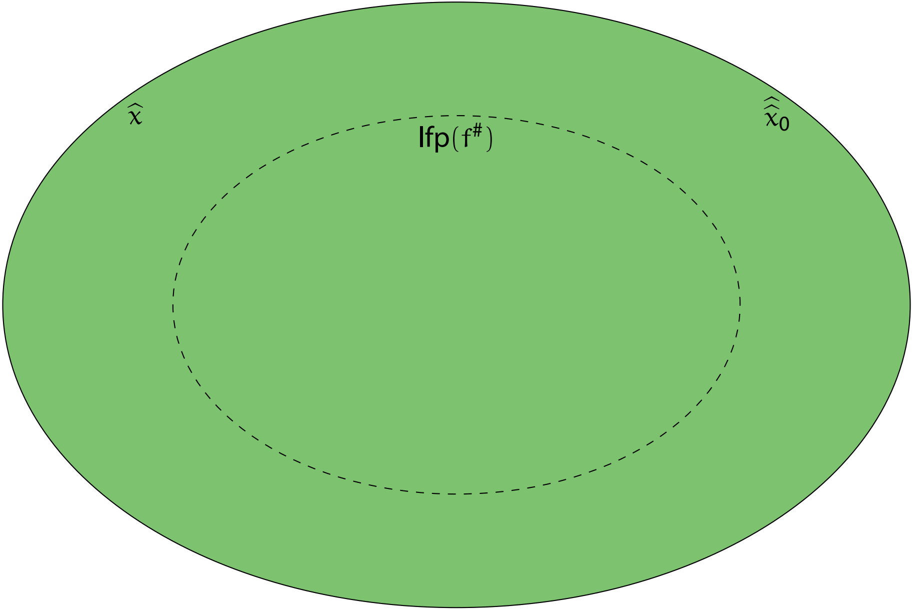

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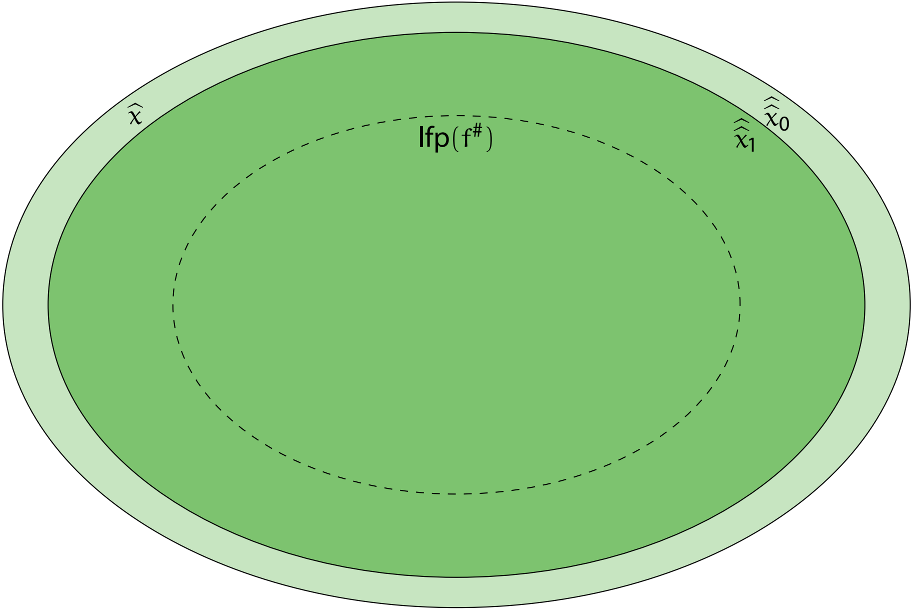

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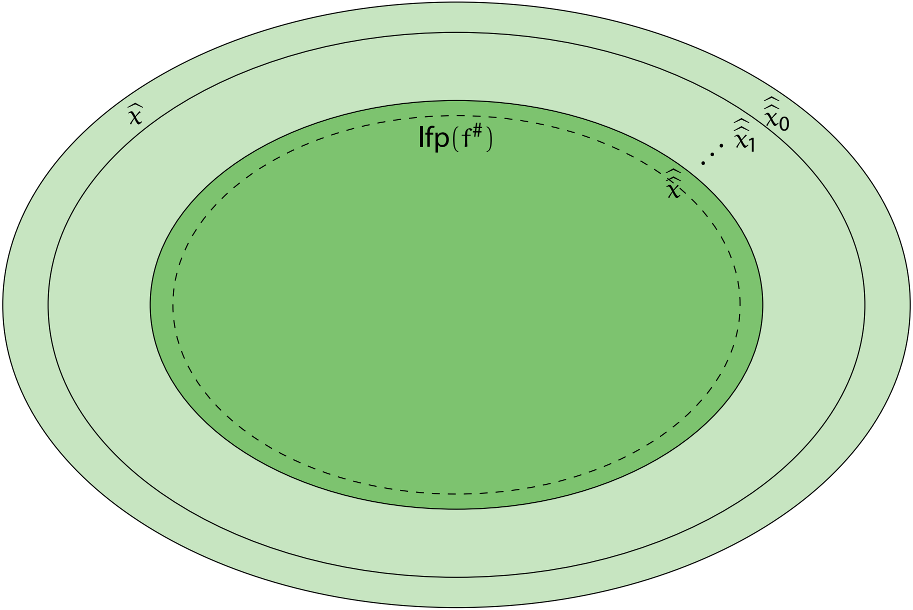

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Widening and Narrowing Example

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    i = 1
    while i <= 100:
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- possible values of `i` at the beginning of the cycle?
- least fixed point of `f`:

$$X = f(X) = (\{1\} \cup \{i + 1 \mid i \in X\}) \cap \{i \in \mathbb{Z} \mid i \leqslant 100\}$$

- with widening, on intervals:

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  - it is not always possible to reach the least fixed point by widening + narrowing

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  - the bounds might not be derivable from the program text
  - more in [CC92]

How to Design Widening and Narrowing? I

- want to define $\nabla, \Delta : L \times L \to L$
- use a finite lattice $\widehat{L}$, such that $L \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} \widehat{L}$:

$$x \nabla y = \gamma(\text{sup}\{\alpha(x), \alpha(y)\})$$
$$x \Delta y = \text{inf}\{x, \gamma(\alpha(y)\}$$

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[l_0, u_0] \vartriangle [l_1, u_1] = [\, & \mathsf{ite}((l_0 \leqslant 0 \leqslant l_1) \vee (l_0 = -\infty), l_1, l_0), \\
& \mathsf{ite}((u_1 \leqslant 0 \leqslant u_0) \vee (l_0 = +\infty), u_1, u_0)]
\end{aligned}
$$

Final notes on Widening and Narrowing

- it is also possible to generalize widening (and narrowing) to work on more previous values
- or all previous values
  - *set widening/set narrowing*

Biblography

Patrick Cousot and Radhia Cousot. "Comparing the Galois connection and widening/narrowing approaches to abstract interpretation". In: *Programming Language Implementation and Logic Programming: 4th International Symposium, PLILP'92 Leuven, Belgium, August 26–28, 1992 Proceedings*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1992, pp. 269–295. DOI: 10.1007/3-540-55844-6_142.