Complementation in Abstract Interpretation

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Reduced Product

Domain Product

$D_1 \times D_2 \rightarrow (D_1, D_2)$
(range) $\times$ {even, odd} $\rightarrow$ ([0,6], odd)

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Reduced Product

- the most precise refinement of the Cartesian product
- let the information flow among the domains to mutually refine them

Galois Connection vs. Insertion

(γ, D, C, α) is an **insertion**:

$$\alpha \circ \gamma = \lambda x.x$$

then:

- α is *surjective*
- γ is *injective*

(Upper) Closure Operator

poset $=$ set $+$ partial order $\leq$
for a poset P, a function $\mathbf{cl} :: P \to P$, which satisfies
$\forall$ elements $x, y \in P$:

- $x \subseteq cl(x)$ *extensive*
- $x \subseteq y \implies cl(x) \subseteq cl(y)$ *increasing*
- $cl(x) = cl(cl(x))$ *idempotent*

Galois Insertion vs. Closure Operators

- D is an abstraction of C if there exist α and γ s.t. (γ, D, C, α) is a Galois insertion
- an abstract domain is "its image in the concrete domain"
- if (γ, D, C, α) is a G.i., then the closure associated with D is the operator $\rho_D = \gamma \circ \alpha$ on C

Closure Operators

$$uco(C)$$

- C - complete lattice
- $uco(C)$ - complete lattice of all closure operators on C

D is an abstraction of C ($C \trianglelefteq D$)
$\Longrightarrow$ $D \cong \rho_D(C)$ for some closure $\rho_D \in uco(C)$

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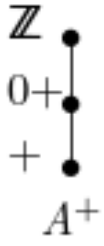
- $uco(C)$ - complete lattice of its abstract interpretations
- .. compared w/ regard to their precision of representation

Reduced Product

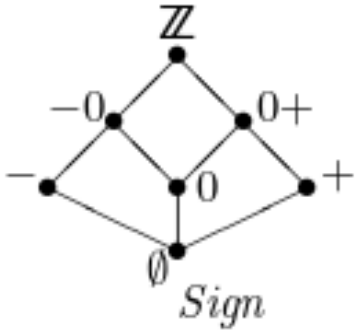
given two abstract domains A and B (abstracting C)

$$A \sqcap B = glb(uco(C))$$



A^+

A^-



Sign

Complementation

- inverse operation to *reduced product* of abstract domains
- A, B - abstract domains, B more abstract than A
  gives as result the most abstract domain $A \sim B$, whose reduced product with B is exactly A
- allows to *decompose* a. domains into simpler factors

Complemented Lattice

- bounded lattice (least element 0, greatest element 1)
- every element a has a complement b,
  s.t. $a \vee b = 1$ and $a \wedge b = 0$

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[1] [15] P. Dwinger. On the closure operators on a complete lattice.
[23] J. Morgado. Note on complemented closure operators of complete lattices.

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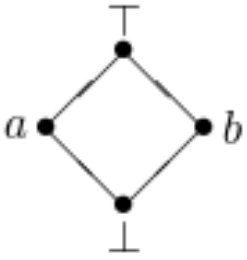
- $uco(C)$ is complemented (equiv. distributive) iff C is a complete well-ordered chain
- too restrictive for abstract interpretation

Complementation

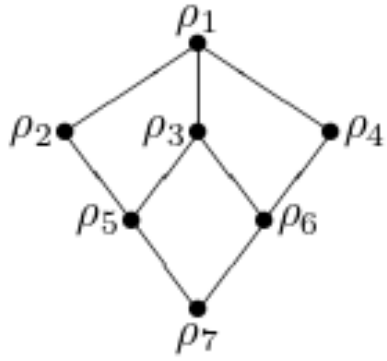
for a lattice L:

- each closure operator ρ is uniquely determined by the set of its fixpoints, which is its image $\rho(L)$
- a set $X \subseteq L$ is the set of fixpoints of a closure operator iff X is a Moore-family of L; i.e. $\top \in X$ and X is meet-closed

Example



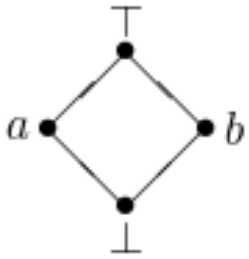
$$\rho_1 = \{\top\}$$
$$\rho_2 = \{\top, a\}$$
$$\rho_3 = \{\top, \bot\}$$
$$\rho_4 = \{\top, b\}$$
$$\rho_5 = \{\top, a, \bot\}$$
$$\rho_6 = \{\top, b, \bot\}$$
$$\rho_7 = \{\top, a, b, \bot\}$$



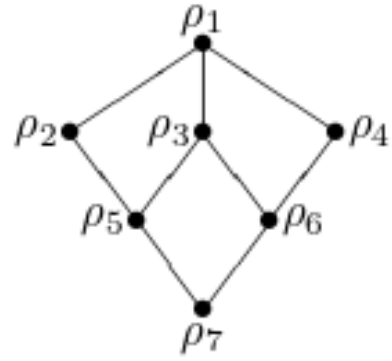
Pseudo-complement

- "calling abusively complement of an abstract interpretation the pseudo-complement of its associated closure operator"
- for a meet-semilattice L and $x \in L$:
  x^* such that $x \wedge x^* = \bot$
  $\forall y \in L.\ (x \wedge y = \bot) \implies (y \preceq x^*)$
- for a complete lattice L:
  $x^* = \vee \{y \in L \mid x \wedge y = \bot\}$

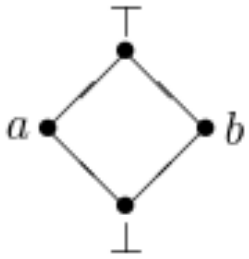
Example



$$\rho_1^* =$$
$$\rho_2^* =$$
$$\rho_3^* =$$
$$\rho_4^* =$$
$$\rho_5^* =$$
$$\rho_6^* =$$
$$\rho_7^* =$$



Example



$$\rho_1^* =$$
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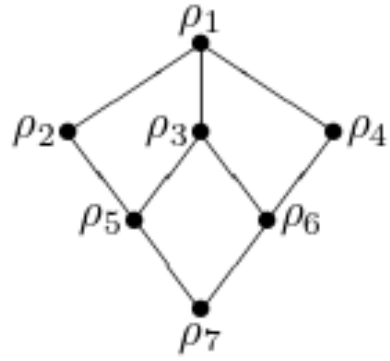

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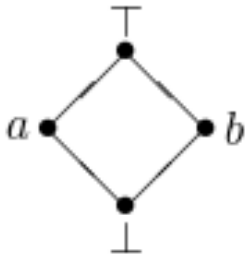

$$\rho_1^* = \rho_7$$
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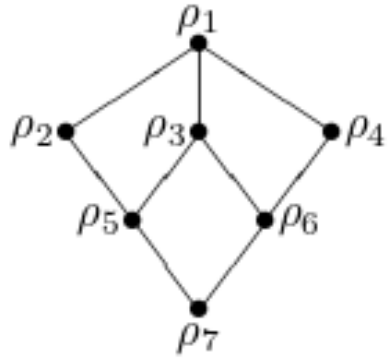


Pseudo-complement

- let B be an abstract domain abstracting another abstract domain A ($A \sqsubseteq B$)
- if A is pseudo-complemented, then the complement of B relative to A is the pseudo-complement B^*
- we denote this complement $A \sim B$
- $(A \sim B) \times B = A$
- shows which properties representable by A are ignored by the abstraction B

Chain-inf Distributivity

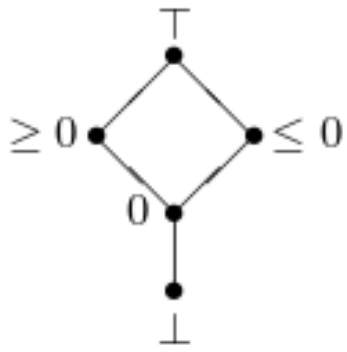
$uco(C)$ is pseudo-complemented $\iff$ C is *chain-inf distributive*
for any chain $C \subseteq L$ and for each $x \in L$

$$x \wedge (\vee\ C) = \vee_{y \in C}\ (x \wedge y)$$

Domain Decomposition

- A *decomposition* of a domain is a tuple $\langle D_i \rangle_{i \in I}$ such that $D = \sqcap_{i \in I} D_i$
- each D_i is an abstraction of D

Domain Decomposition



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$$\rho_{14} = D$$

Domain Decomposition

$\rho_1 = \{\top\}$; $\rho_2 = \{\top, \geq 0\}$; $\rho_3 = \{\top, 0\}$; $\rho_4 = \{\top, \bot\}$; $\rho_5 = \{\top, \leq 0\}$;
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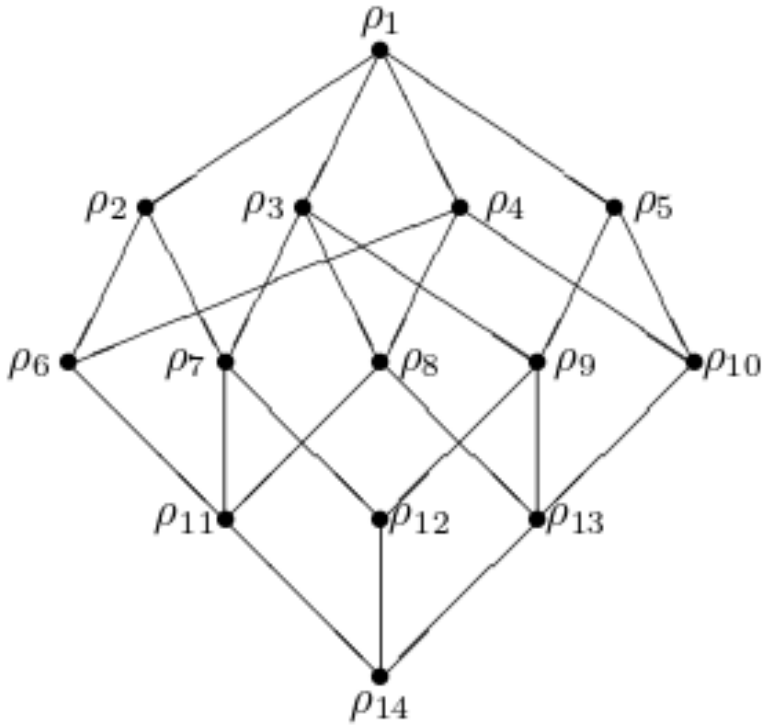

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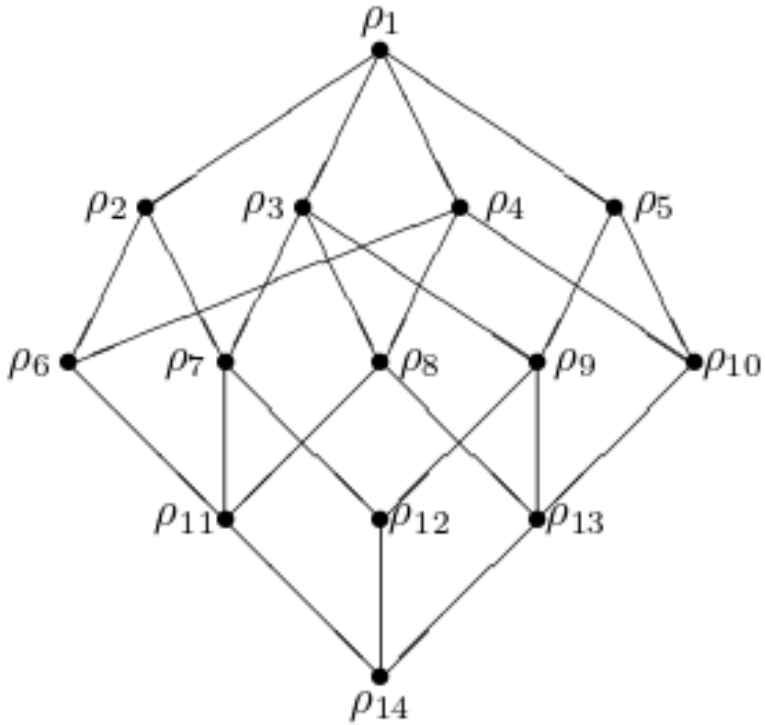

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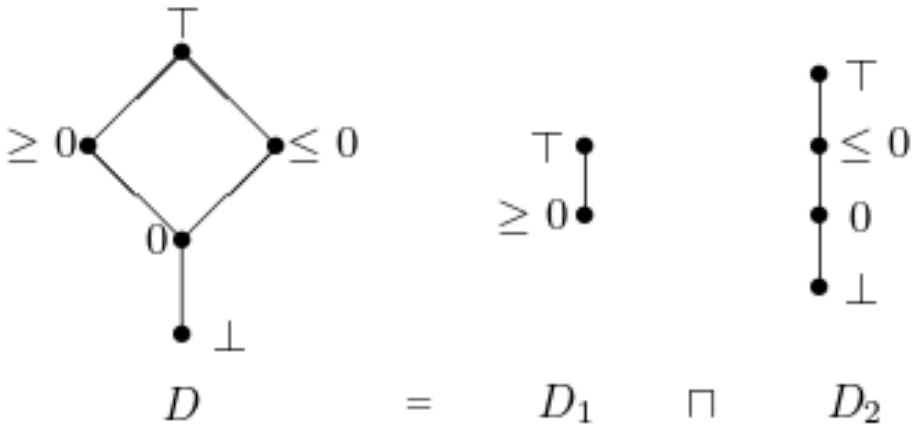
Domain Decomposition



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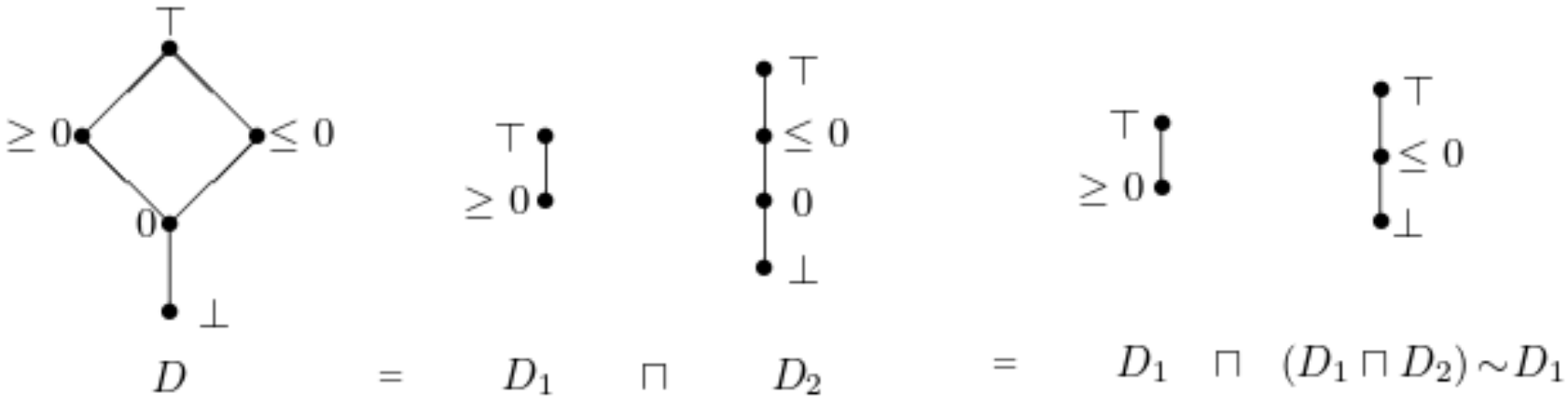

Sharing

- complementation used to characterize what is left when we eliminate from it the information useful for ground-dependency analysis (expressed by a more abstract domain *Def*)
- the complement of *Def* w/ respect to *Sharing*, called $Sharing^{+}$, captures precisely variable aliasing and no ground-dependency information